

Rough contact mechanics for graded bulk rheology: The role of small-scale wavelengths on rubber friction

Michele Scaraggi^{1*} and Davide Comingio¹

¹*DII, Università del Salento, 73100 Monteroni-Lecce, Italy*

(Dated: October 15, 2015)

We present a numerical model for the prediction of the rough contact mechanics of a viscoelastic block, with graded rheology, in steady sliding contact with a randomly rough rigid surface. In particular, we derive the effective surface response of a stepwise or continuously-graded block in the Fourier domain, which is then embedded in a Fourier-based residuals molecular dynamic formulation of the contact mechanics. Finally we discuss on the role of small-scale wavelengths on rubber friction and contact area, and we demonstrate that the rough contact mechanics exhibits effective interface properties which converge to asymptotes upon increase of the small-scale roughness content, when a realistic rheology of the confinement is taken into account.

Keywords: Sliding friction, rubber friction, wear modified surface layer, coating, graded viscoelasticity, rough contact mechanics, roughness.

* Corresponding author; michele.scaraggi@unisalento.it

I. INTRODUCTION

The rough contact mechanics of solids exhibiting graded rheology at confinement is of major interest for both technological (e.g. seals, rubber friction, tribology of protective coatings) and biological (tissue engineering, bio-lubrication in the bone cartilage contact) applications, to cite few. On the theoretical side, the graded functionalization of surfaces has been long believed to provide the short distance cutoff, in the range of available contact roughness wavelengths (usually extending to the atomic distance, given the fractal nature of the real surfaces), as the physical threshold over which smaller asperities do not contribute to the effective properties of the contact. As an example, in a generic dry rubber contact, surface contamination[2, 6, 14] as well as a *thin skin*[2, 9] on the rubber surface, e.g. generated as a consequence of a steady-state wear, can strongly alter the effective surface properties of the confinement. Typically, this contamination layer will avoid the smallest roughness asperities to contribute to the energy dissipation, thus reducing the friction force with respect to the ideal value. Nevertheless, literature provides no strong theoretical support to this aspect, mainly due to the lack of (both analytical and numerical) mean field modelling of rough contact mechanics in presence of graded rheology at confinement, apart few investigations[4, 10]. In particular, in Ref. [10], a GW-like (many-asperities) formulation of the rough contact mechanics under some simplified graded-response assumptions is presented, whereas in Ref. [4] the Persson's multiscale contact theory[2] has been extended and the contact area calculated in the case of an elastic coating bonded onto an elastic half space. In particular, in [4] the $M_{zz}(\omega, \mathbf{q})$ function, which provides in the Fourier domain the (out-of-plane) surface displacement response as a function of the contact pressure field, is derived for the case of a coating bonded onto a half space, under the assumption of frequency-independent Poisson coefficient.

In this work we will make use of the field decomposition suggested in [2] to determine the effective $M_{zz}(\omega, \mathbf{q})$ (in the Fourier domain) function for the more general case of a stepwise or continuously-graded block. The Poisson coefficient for the generic layer is assumed frequency-dependent, which makes the theory of more general applicability. This effective surface response will be then implemented in a Fourier-based residuals molecular dynamic (RMD) formulation of the contact mechanics[11–13], however, the same function might be easily embedded in the Persson's mean field analytical contact model as well. The RMD model will then be adopted to a focussed investigation of the role of small-scale roughness wavelengths on the rubber friction and contact area. We numerically show that the rough contact exhibits effective interface properties which converge to asymptotes upon increase of the small-scale roughness content, when a realistic rheology of the confinement, which includes a graded rheology, is taken into account. For the rubber contact case a graded rheology has been recently experimentally shown in Ref. [9], where the authors clearly demonstrate the existence of a modified surface layer with strongly different properties than the bulk, inspiring indeed our theoretical investigation.

The manuscript is organized as follows. In Sec. II we summarize the BEM (boundary element method) numerical scheme adopted for the investigation of a steady-sliding rough interaction characterized by arbitrary rheological properties, whereas in Sec. III we more specifically focus on the calculation of the surface displacement Green's function, in the Fourier domain $M_{zz}(\mathbf{q}, \omega)$, for a generic block with stepwise graded (isotropic) viscoelastic rheology. In Sec. IV we apply the numerical model to the investigation of the role of the (wear) modified surface layer (MSL) on the rubber friction and contact area for the simplest case of a rubber bulk covered by an elastic coating, in steady sliding contact onto a randomly rough rigid surface. Finally, in Sec. V we more generally discuss on the role of the graded bulk rheology in rough contact mechanics, whereas in Sec. VI the conclusions follow. In Appendix A we solve the Navier's equation for a homogeneously-viscoelastic infinitely-wide slab (of finite thickness) in the quasi-static deformation dynamics. Analytical relations are derived for the surface response of coated bulks in the most general case of non-constant (in the frequency domain) Poisson's ratio. We show that, even for the simplest case of coating on bulk, the adoption of constant Poisson ratio (i.e. independent from frequency) lead to qualitatively different results with respect to the more general case of frequency dependent Poisson ratio, in a range of roughness wavelengths. In Appendix B the surface response for the general case of continuously-graded viscoelastic rheology is formulated in term of a set of non-linear differential equations, which is solved in the two representative cases of linear and sinusoidal variation of the confined elastic properties of a bulk. The results are then compared with the predictions of the stepwise-graded theory (Sec. III) as applied to a discretized version of the confinement (at different numbers of divisions in sub-layers) showing, for the sinusoidal variation, that the number of layers needed for the stepwise-graded surface response to converge to the continuously-graded predictions can be (relatively) very large.

II. SUMMARY OF THE NUMERICAL SCHEME FOR A STEADY-SLIDING ROUGH INTERACTION

We consider the case of a rigid, *periodically-rough* surface (of L_0 periodic length, in both x - and y -direction, with small wavelength cut-off frequency $q_0 = 2\pi/L_0$) in steady sliding adhesionless contact with a graded body characterized by linear rheology, under isothermal and frictionless conditions. We assume the small deformation regime to apply,

as well as a small square slope roughness $h(\mathbf{x})$, with $m_2 = \langle \nabla h(\mathbf{x})^2 \rangle \ll 1$ ($\langle h \rangle = 0$). In Fig. 1 we show a schematic

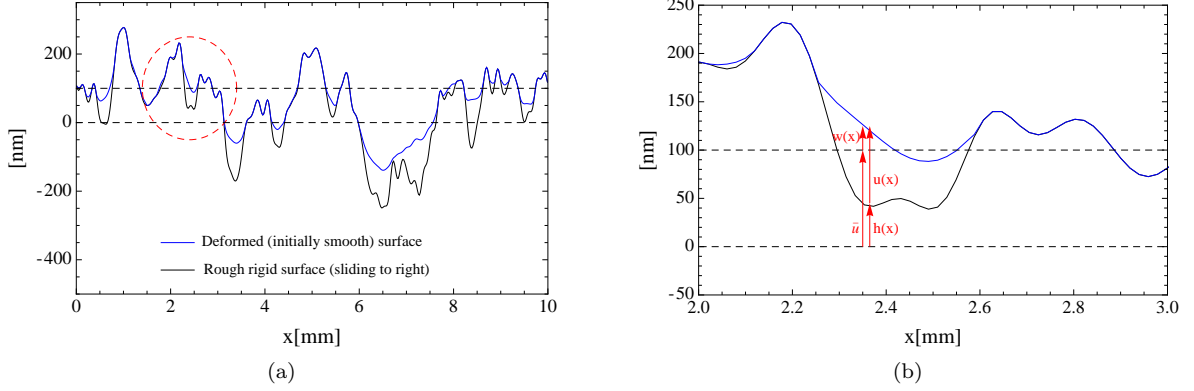


Figure 1. a) Cross section of a generic contact interface. b) Magnified view of the encircled area (in the left figure), with indication of the gap equation (1) terms. Schematics.

representation of the contact interface, in a reference moving with the rough sliding surface. In such a reference, the local interfacial separation $u(\mathbf{x})$ can be agreed to be:

$$u(\mathbf{x}) = \bar{u} + w(\mathbf{x}) - h(\mathbf{x}), \quad (1)$$

where \bar{u} is the average interfacial separation, $w(\mathbf{x})$ the surface out-of-average-plane displacement and $h(\mathbf{x})$ the surface roughness, with $\langle w(\mathbf{x}) \rangle = \langle h(\mathbf{x}) \rangle = 0$. We can define the following:

$$w(\mathbf{q}) = (2\pi)^{-2} \int d^2\mathbf{x} w(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}$$

and

$$\sigma(\mathbf{q}) = (2\pi)^{-2} \int d^2\mathbf{x} \sigma(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}},$$

where $\sigma(\mathbf{x}) = \Delta\sigma(\mathbf{x}) + \sigma_0$ is the distribution of interfacial pressures [$\sigma_0 = \langle \sigma(\mathbf{x}) \rangle$ is the average contact pressure] in the moving reference. Following the discussion reported in Sec. III, $w(\mathbf{x})$ can be related to $\sigma(\mathbf{x})$ through a simple equation in the Fourier space

$$w(\mathbf{q}) = M_{zz}(\mathbf{q}, \omega = \mathbf{q} \cdot \mathbf{v}) \sigma(\mathbf{q}), \quad (2)$$

where $M_{zz}(\mathbf{q}, \omega)$ is the complex surface response of the block in the frequency domain, and \mathbf{v} the sliding velocity [in this work $\mathbf{v} = (v, 0)$ without any loss of generality]. $M_{zz}(\mathbf{q}, \omega)$ depends on the rheological and geometrical properties of the block, and its formulation will be specifically presented in Sec. III. We observe that in the simplest case of bulk viscoelastic contact with frequency-independent Poisson ratio, $M_{zz}(\mathbf{q}, \omega) = 2/[|\mathbf{q}| E_r(\omega)]$, where $E_r(\omega) = E(\omega)/(1 - \nu^2)$ is the frequency-dependent (complex) reduced Young's modulus, ν is the Poisson ratio. In this work the assumption of constant Poisson ratio will not be adopted (unless differently explicated), thus the theory developed hereinafter is of more general applicability, e.g. it can be easily adapted to existing mean field contact mechanics formulations as well. (2) is obtained by considering that the stress in the fixed reference $\sigma(\mathbf{q}, \omega) = \sigma(\mathbf{q}) \delta(\omega - \mathbf{q} \cdot \mathbf{v}_0)$ is related to the displacement in the fixed reference $w(\mathbf{q}, \omega) = w(\mathbf{q}) \delta(\omega - \mathbf{q} \cdot \mathbf{v}_0)$ through the constitutive relationship $w(\mathbf{q}, \omega) = M_{zz}(\mathbf{q}, \omega) \sigma(\mathbf{q}, \omega)$ (see Sec. III) resulting, after integration over ω , in 2.

Finally, the relation between separation $u(\mathbf{x})$ and interaction pressure $\sigma(\mathbf{x})$ is calculated within the Derjaguin's approximation [5], and it can be written in term of a generic interaction law [11, 13]:

$$\sigma(u) = f(u). \quad (3)$$

In this work we have adopted the (integrated) repulsive term of the L-J potential in (3) to simulate the adhesionless interaction. However, the theory can be easily extended to other interface laws [11]. (1), (2) and (3) are discretized on a regular square mesh of grid size δ in term of a residuals molecular dynamics process [11, 12], resulting in the

following set of equations:

$$L_{ij} = -u_{ij} + (\bar{u} + w_{ij} - h_{ij}) \quad (4)$$

$$\sigma_{ij} = f(u_{ij}) \quad (5)$$

$$\sigma_{ij} \rightarrow \Delta \sigma(q_{hk}) = M_{zz}^{-1} w(q_{hk}) \rightarrow w(x_{ij}) \quad (6)$$

where L_{ij} is the generic residual (related to the generic iterative solution u_{ij}). (4) are solved in u_{ij} though a Verlet intergration scheme, whereas the solution accuracy is set by requiring

$$\begin{aligned} \langle L_{ij}^2 / u_{ij}^2 \rangle^{1/2} &< \varepsilon_L \\ \langle [(u_{ij}^n - u_{ij}^{n-1}) / u_{ij}^{n-1}]^2 \rangle^{1/2} &< \varepsilon_u, \end{aligned}$$

where both errors are typically of order 10^{-4} .

Among the mean physical quantities which can be extracted from the solution fields, the one of particular relevance for this work is the micro-rolling friction coefficient $\mu_r = F_r / F_N$, where the micro-rolling force F_r reads

$$F_r = |\mathbf{v}|^{-1} \int_{A_0} d^2x \, \sigma(\mathbf{x}) [\nabla h(\mathbf{x}) \cdot \mathbf{v}],$$

and the normal load $F_N = \int_{A_0} d^2x \, \sigma(\mathbf{x})$.

III. $M_{zz}(\mathbf{q}, \omega)$ FOR A BULK WITH STEPWISE-GRADED VISCOELASTIC RHEOLOGY

In this section we will show how to calculate $M_{zz}(\mathbf{q}, \omega)$ for a stepwise-graded viscoelastic composite. The case of continuously-graded viscoelastic composite will be discussed in Appendix B. In particular, we first consider the case

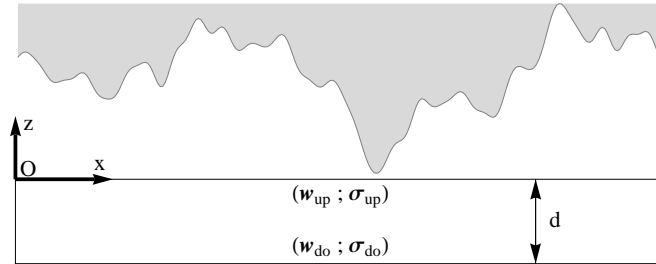


Figure 2. Schematic of a infinitely-wide slab of finite thickness d , characterized by a linear viscoelastic rheology. σ_{up} (σ_{do}) and w_{up} (w_{do}) are, respectively, the stress and displacement fields on the top $z = 0$ (bottom $z = -d$) surface.

of a linearly-viscoelastic infinitely-wide slab of thickness d , see Fig. 2. By considering the following Fourier transform ($t \rightarrow \omega$ and $\mathbf{x} \rightarrow \mathbf{q}$)

$$\begin{aligned} \mathbf{w}(\mathbf{q}, z, \omega) &= (2\pi)^{-3} \int dt \int d^2x \, \mathbf{w}(\mathbf{x}, z, t) e^{-i(\mathbf{q} \cdot \mathbf{x} - \omega t)} \\ \mu(\omega) &= \int dt \, \mu(t) e^{-i(-\omega t)} \end{aligned}$$

and, inversely,

$$\begin{aligned} \mu(t) &= (2\pi)^{-1} \int d\omega \, e^{i(-\omega t)} \mu(\omega) \\ \mathbf{w}(\mathbf{x}, z, t) &= \int dt \int d^2x \, \mathbf{w}(\mathbf{q}, \omega) e^{i(\mathbf{q} \cdot \mathbf{x} - \omega t)}, \end{aligned}$$

the relation between the stress and displacement fields on the top ($z = 0$) and bottom surface ($z = -d$), in the limit of quasi-static interaction [i.e. $\omega/(qc) = v/c \ll 1$, see Appendix A, where c is the generic sound speed] reads in matrix form

$$\begin{bmatrix} \boldsymbol{\sigma}_{\text{up}}/[E_r(\omega)q/2] \\ \mathbf{w}_{\text{up}} \end{bmatrix} = \cosh(qd) \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_{\text{do}}/[E_r(\omega)q/2] \\ \mathbf{w}_{\text{do}} \end{bmatrix}, \quad (7)$$

where $\mathbf{M}_j[\mathbf{q}d, \nu(\omega)]$ is a 3 by 3 matrix. $\boldsymbol{\sigma}_{\text{up}}$ ($\boldsymbol{\sigma}_{\text{do}}$) and \mathbf{w}_{up} (\mathbf{w}_{do}) are, respectively, the stress and displacement fields on the top (bottom) surface, see Fig. 2. \mathbf{M}_j are determined in Appendix A for the most general case of frequency-dependent Poisson ratio $\nu(\omega)$, as well as reported for the limiting case of constant ν . (7) can be conveniently rephrased depending on the adopted boundary conditions (BCs) on the bottom surface ($z = -d$), e.g.

$$\mathbf{w}_{\text{up}}(\mathbf{q}, \omega) = \mathbf{M}_3 \mathbf{M}_1^{-1} \frac{\boldsymbol{\sigma}_{\text{up}}(\mathbf{q}, \omega)}{E_r(\omega)q/2} + \cosh(qd) [\mathbf{M}_4 - \mathbf{M}_3 \mathbf{M}_1^{-1} \mathbf{M}_2] \mathbf{w}_{\text{do}}(\mathbf{q}, \omega) \quad (8)$$

or

$$\mathbf{w}_{\text{up}}(\mathbf{q}, \omega) = \mathbf{M}_4 \mathbf{M}_2^{-1} \frac{\boldsymbol{\sigma}_{\text{up}}(\mathbf{q}, \omega)}{E_r(\omega)q/2} + \cosh(qd) [\mathbf{M}_3 - \mathbf{M}_4 \mathbf{M}_2^{-1} \mathbf{M}_1] \frac{\boldsymbol{\sigma}_{\text{do}}(\mathbf{q}, \omega)}{E_r(\omega)q/2}. \quad (9)$$

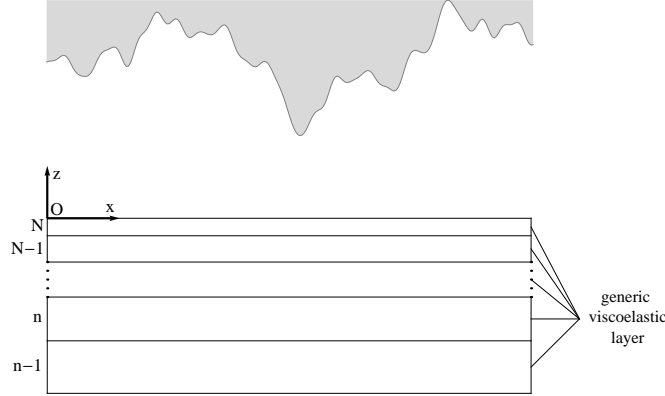


Figure 3. Schematic of an infinitely-wide slab of finite thickness $d = \sum_j d_j$, characterized by a step-wise graded linear viscoelastic rheology.

Now, in Fig. 3 we show the schematic of the generic composite slab with a step-wise graded rheology, with $j = 1..n..N$ bonded layers. We first assume the generic layer ($n - 1$) to be described by the general stress-displacement relation

$$\mathbf{w}_{\text{up}} = [\mathbf{M}]^{(n-1)} \frac{\boldsymbol{\sigma}_{\text{up}}}{[E_r(\omega)]^{(n-1)} q/2},$$

where \mathbf{M} is a 3 by 3 matrix. Imposing the continuity of stress and displacement between layer ($n - 1$) and (n), and by using (7), we get for the layer (n)

$$\mathbf{w}_{\text{up}} = \left[\mathbf{M}_3 + \frac{E_r(\omega)}{[E_r(\omega)]^{n-1}} \mathbf{M}_4 [\mathbf{M}]^{n-1} \right] \left[\mathbf{M}_1 + \frac{E_r(\omega)}{[E_r(\omega)]^{n-1}} \mathbf{M}_2 [\mathbf{M}]^{n-1} \right]^{-1} \frac{\boldsymbol{\sigma}_{\text{up}}}{E_r(\omega)q/2},$$

where the index (n) has been dropped for simplicity. Thus \mathbf{M} for the layer (n) reads

$$[\mathbf{M}]^{(n)} = \left[\mathbf{M}_3 + \frac{E_r(\omega)}{[E_r(\omega)]^{(n-1)}} \mathbf{M}_4 [\mathbf{M}]^{(n-1)} \right] \left[\mathbf{M}_1 + \frac{E_r(\omega)}{[E_r(\omega)]^{(n-1)}} \mathbf{M}_2 [\mathbf{M}]^{(n-1)} \right]^{-1}, \quad (10)$$

with again

$$\mathbf{w}_{\text{up}} = [\mathbf{M}]^{(n)} \frac{\boldsymbol{\sigma}_{\text{up}}}{[E_r(\omega)]^{(n)} q/2}.$$

(10) shows that the surface response of a stepwise-graded composite can be easily determined with a recursive calculation.

Finally, for the stepwise graded composite with N -layers

$$M_{zz}(\mathbf{q}, \omega) = \frac{2 [\mathbf{M}(\mathbf{q}, \omega)]_{3,3}^{(N)}}{q [E_r(\omega)]^{(N)}}, \quad (11)$$

where $[\mathbf{M}]^{(0)}$ [innermost layer, needed to initialize (10)] is obtained from (8) or 9, depending on the adopted BCs $\mathbf{w}_{\text{do}}(\mathbf{q}, \omega) = 0$ (thus $[\mathbf{M}]^{(0)} = \mathbf{M}_3 \mathbf{M}_1^{-1}$) or $\sigma_{\text{do}}(\mathbf{q}, \omega)$ (thus $[\mathbf{M}]^{(0)} = \mathbf{M}_4 \mathbf{M}_2^{-1}$), for $q \neq 0$. In the simplest case of a bulky (0)-layer (corresponding to a half space, i.e. $d \rightarrow \infty$), $[\mathbf{M}]^{(0)} = \mathbf{M}_3 \mathbf{M}_1^{-1} = \mathbf{M}_4 \mathbf{M}_2^{-1}$ reads

$$[\mathbf{M}]^{(0)} = \begin{bmatrix} 1 + \frac{\nu}{p} \frac{q_y^2}{q^2} & -\frac{\nu}{p} \frac{q_x q_y}{q^2} & i \frac{2\nu-1}{2p} \frac{q_x}{q} \\ -\frac{\nu}{p} \frac{q_x q_y}{q^2} & 1 + \frac{\nu}{p} \frac{q_x^2}{q^2} & i \frac{2\nu-1}{2p} \frac{q_y}{q} \\ i \frac{1-4\nu(1-\nu_0)}{1-2\nu_0} \frac{q_x}{2pq} & i \frac{1-4\nu(1-\nu_0)}{1-2\nu_0} \frac{q_y}{2pq} & \frac{p_0}{p} \frac{1-2\nu}{1-2\nu_0} \end{bmatrix}. \quad (12)$$

In (12) $p = 1 - \nu(\omega)$ and $p_0 = 1 - \nu_0$, where $\nu_0 = \nu(\omega = 0)$ is the Poisson coefficient in the rubbery regime. Observe that for the most general case ($p \neq p_0$) $[\mathbf{M}]_{3,3}^{(0)}$ in (12) is not equal to 1, as (indirectly) expected from the theory of the elastic-viscoelastic correspondance. This also suggests that our model can easily overtake the restrictions imposed by the adoption of the elastic-viscoelastic correspondance principle (frequency-independent Poisson ratio) to the rubber rheological characteristics which can be modelled. In the simplest case where $\nu(\omega) = \nu = \nu_0$, 12 simplifies to the well known

$$[\mathbf{M}]^{(0)} = \begin{bmatrix} 1 + \frac{\nu}{p} \frac{q_y^2}{q^2} & -\frac{\nu}{p} \frac{q_x q_y}{q^2} & i \frac{2\nu-1}{2p} \frac{q_x}{q} \\ -\frac{\nu}{p} \frac{q_x q_y}{q^2} & 1 + \frac{\nu}{p} \frac{q_x^2}{q^2} & i \frac{2\nu-1}{2p} \frac{q_y}{q} \\ i \frac{1-2\nu}{2p} \frac{q_x}{q} & i \frac{1-2\nu}{2p} \frac{q_y}{q} & 1 \end{bmatrix}. \quad (13)$$

Finally, the linear viscoelastic complex modulus (which can be measured recurring to standard techniques [1]) $E(\omega)$ can be related to the creep spectrum through a Prony series [1, 7, 8], obtaining

$$\frac{1}{E(\omega)} \approx \frac{1}{E_\infty} + \sum_{j=1}^N \frac{H(\tau_j)}{1 - i\omega\tau_j} \quad (14)$$

where N is the number of relaxation times, $H(\tau_j)$ the discrete creep function, and τ_j the generic relaxation time. E_∞ is the elastic modulus in the glassy regime. In Fig. 4 we show the real part of the (dimensionless) reduced viscoelastic

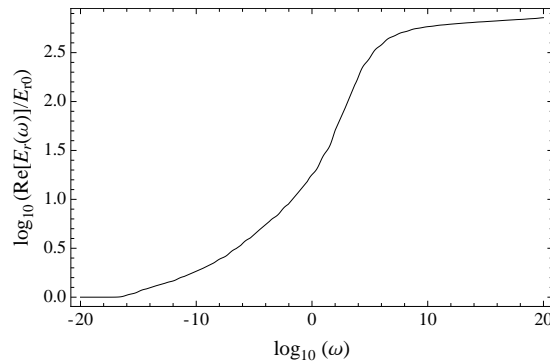


Figure 4. Real part of the (dimensionless) reduced viscoelastic modulus $E_r(\omega)/E_{r0}$ adopted in this work, as a function of the frequency (in s^{-1}), in \log_{10} - \log_{10} . For a car tire rubber compound at a low external temperature.

modulus $E_r(\omega)/E_{r0}$ adopted in this work, as a function of the frequency (in s^{-1}), in a \log_{10} - \log_{10} scale. The adopted viscoelastic modulus corresponds to a car tire tread-block compound under low operating temperatures (see e.g. [13]).

IV. NUMERICAL RESULTS

In this section we investigate for the first time the rough contact mechanics of a rubber block (see Fig. 4 for the rheological properties) covered by a surface layer with modified rheological properties (with respect to the bulk), with particular focus to hysteretic friction (i.e. micro-rolling friction) and contact area (directly related to the adhesive contribution to friction). We observe that a modified surface layer (MSL) of thickness order $\approx 1\mu\text{m}$ usually occurs as a consequence of rubber wear [9] in e.g. tire tread-road contacts, or in dynamic rubber seals. The MSL thickness is clearly expected to introduce a high frequency (physical) cut-off to the roughness spectral content which can be probed by the bulk, making the contact mechanics unaffected by the small-wavelength roughness regime beyond such a threshold. We further observe that without such a physical cut-off mechanism, the hysteretic friction (normalized contact area) increases (decreases) theoretically unbounded in an ideal randomly rough interaction [2, 13], thus (classically) making the quantitative prediction of friction and contact area to be strongly dependent on the arbitrary (or fitted) choice of such threshold parameter.

The following results are obtained by applying the numerical model developed in Sec. II and III to the case of an elastic coating bonded onto a viscoelastic half space, in steady sliding contact with a rigid isotropically-rough surface. The bulk is characterized by the complex reduced viscoelastic modulus $E_r(\omega)$ of the tread rubber compound reported in Sec. III, whereas the elastic coating is assumed here to be characterized by the reduced Young's modulus $E_{r0} = E_r(\omega = 0)$, i.e. given by the rubber relaxed elastic modulus (in qualitative agreement with the experimental observations [9])¹.

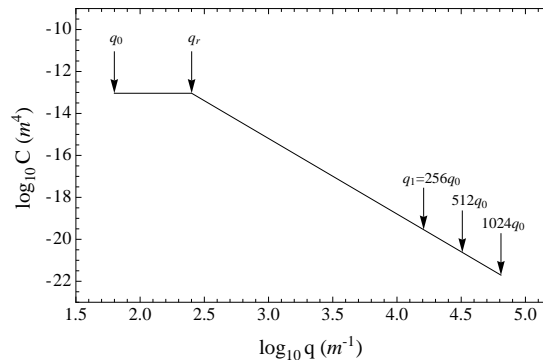


Figure 5. Roughness power spectral density adopted in the present study, in \log_{10} - \log_{10} scale. The power spectra have a low wave vector cut-off for $q_0 = 0.63 \cdot 10^2 \text{ m}^{-1}$, and a roll-off for $q_r = 4q_0$. For $q > q_r$ the power spectra correspond to self-affine fractal surfaces with Hurst exponent $H = 0.8$. We consider three cases where the large wave vector cut-off is $q_1 = 256q_0$, $512q_0$, and $1024q_0$ (corresponding to a root mean square slope $s_{\text{rms}} = \langle |\nabla h|^2 \rangle^{1/2} = 0.077$, 0.095 , and 0.11 , respectively). The root mean square roughness, which is mostly determined by large wavelengths content, is similar for all cases to $h_{\text{rms}} = 0.16 \text{ mm}$. All calculations have been performed with $n = 8$ divisions at the small wavelength $\lambda_1 = 2\pi/q_1$.

Figure 5 shows the generic roughness power spectral density adopted in the present study. The power spectra have a low wave vector cut-off for $q_0 = 0.63 \cdot 10^2 \text{ m}^{-1}$, and a roll-off for $q_r = 4q_0$. For $q > q_r$ the power spectra correspond to self-affine fractal surfaces with Hurst exponent $H = 0.8$ (related to the fractal dimension $D_F = 3 - H$)². We consider three cases where the large wave vector cut-off is $q_1 = 256q_0$, $512q_0$, and $1024q_0$ (corresponding to a root mean square slope $s_{\text{rms}} = \langle |\nabla h|^2 \rangle^{1/2} = 0.077$, 0.095 , and 0.11 , respectively). The root mean square roughness, which is mostly determined by large wavelength content, is similar for all cases to $h_{\text{rms}} = 0.16 \text{ mm}$.

¹We stress that the graded rheological formulation we have developed can be applied to any, continuous or stepwise, formulation of the composite, whose rheological characteristics as a function of the bulk depth can be obtained e.g. through a sub-surface differential measurements of mechanical properties. However, within the theoretical purposes of this contribution, which has been intentionally limited to the fundamental understanding of graded rheology (as e.g. induced by wear-driven MSL formation) on the rough contact mechanics, we have numerically simulated a composite formulation described by the smallest set of interaction parameters (e.g. elastic coating onto rubber bulk), yet complete enough to capture the basic physics occurring in the rough interaction between graded solids.

²We stress that whilst roughness self-affine characteristics are often found in several man- and nature-made surfaces [3], the self-affine behaviour is here adopted only for convenience (as discussed before), in order to reduce the number of parameters characterizing the contact interface. However, there is no particular limitation in the deterministic or statistically complexity of the rough surfaces to be simulated.

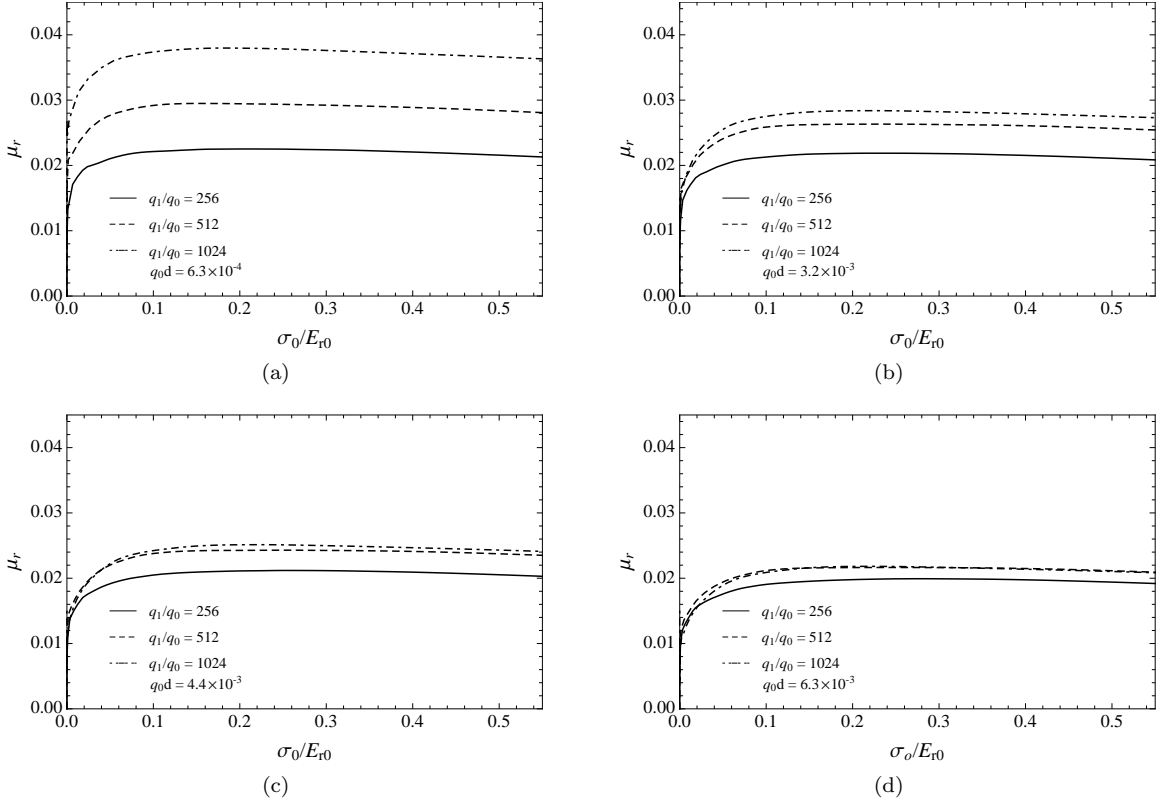


Figure 6. Micro-rolling friction μ_r as a function of the dimensionless contact pressure σ_0/E_{r0} , for $q_1/q_0 = 2^8, 2^9$ and 2^{10} . The sliding velocity v is set to 0.1 m/s. From (a) to (d) $q_0d \cdot 10^3 = 0.63, 3.2, 4.4, 6.3$.

In Fig 6 and 7 we show, respectively, the micro-rolling friction μ_r and the contact area A_c/A_0 as a function of the dimensionless contact pressure σ_0/E_{r0} , for $q_1/q_0 = 2^8, 2^9$ and 2^{10} . The sliding velocity v is set to 0.1 m/s. From (a) to (d) the coating thickness is increased from $q_0d = 0.63 \cdot 10^{-4}$ to $6.3 \cdot 10^{-3}$ (i.e. $d = 10$ to $100 \mu\text{m}$ for our system). In particular, Figs. 6(a) and 7(a) show, respectively, μ_r and A_c/A_0 for the thinner coating ($d = 10 \mu\text{m}$). We observe that increasing the roughness high frequency content determines an increase (decrease) of the hysteretic friction (true contact area). This could be expected from classical mean field theories when observing that the coating is thin enough to allow the whole range of asperities, down to the smallest wavelengths (i.e. to $q_1/q_0 = 2^{10}$), to probe the rubber bulk and, therefore, to effectively contribute generating the stored (responsible for the contact area) and dissipated interfacial energy. Moreover, in accordance with classical results [2, 13], even a small increase in the roughness spectral content in the high-frequency regime non-negligibly affects both friction and contact area, as due to the strong dependence of such physical quantities on the s_{rms} (which increases from 0.07 to 0.11 from $q_1/q_0 = 2^8$ to 2^{10}). Figs. 6(d) and 7(d) show, respectively, μ_r and A_c/A_0 for the thicker coating ($d = 100 \mu\text{m}$). In this case the contact prediction for $q_1/q_0 = 2^9$ and 2^{10} overlap, i.e. an asymptotic friction and contact area are obtained for $q_1/q_0 = 2^9$ *in the entire set of investigated contact pressures*. Interestingly, such asymptotes markedly differ from the corresponding curves of Figs. 6(a) and 7(a) and are now closer to the $q_1/q_0 = 2^8$ curve; instead, the predictions at $q_1/q_0 = 2^8$ are almost unaffected by the coating thickness. This can be easily justified with the following arguments. In particular, by increasing the coating size the smallest roughness wavelengths are no more able to probe the viscoelastic bulk, hence the hysteretic friction is unaffected by the smallest asperities resulting into a negligible dissipation increase with respect to the $q_1/q_0 = 2^8$ roughness case. Furthermore, given the soft rheological characteristics of the elastic coating, the smallest asperities are in full contact with the substrate (at increasing coating thickness), with no relevant effect in term of contact area reduction, even at small contact area values (i.e. in the linear regime³). Indeed the smallest wavelength asperities probe a locally soft solid, which is even not subjected to the sliding induced viscoelastic

³Note that in the linear contact regime, the average effective contact pressure $\bar{\sigma}$ [$\bar{\sigma}(q_1) = \sigma_0 A_0 / A_c(q_1)$, where $A_c(q_1)$ is the true contact area when the power spectral density contains roughness down to q_1 wavenumber], which is responsible for the local asperity contact condition, is a magnification-only dependent value, i.e. given $A_c(q_1) \approx A_0 k(q_1) m_2^{-1/2}(q_1) \sigma_0 / E_r(vq_1)$, one obtains $\bar{\sigma}(q_1) k(q_1) m_2^{-1/2}(q_1) / E_r(vq_1) \approx 1$. Thus, locally, the asperities will be in partial or full contact depending on the actual value of $k(q_1)$.

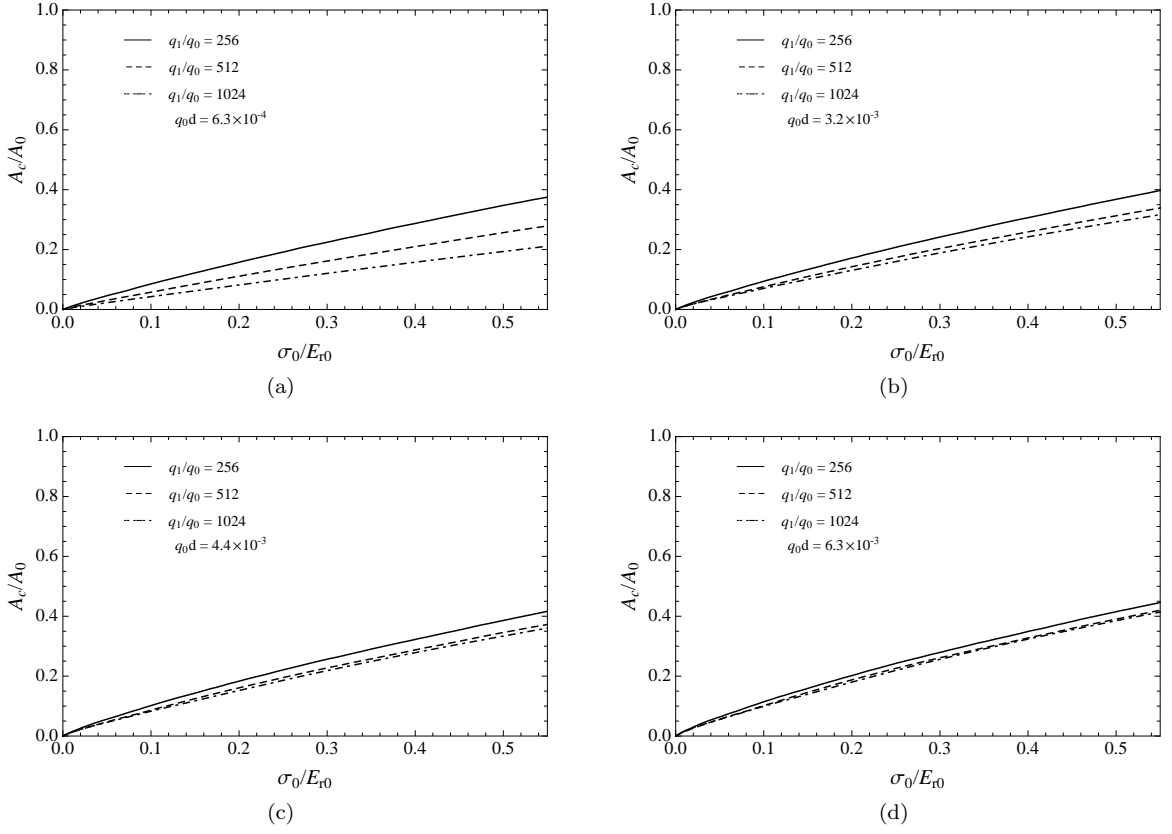


Figure 7. Normalized (projected) contact area A_c/A_0 as a function of the dimensionless contact pressure σ_0/E_{r0} , for $q_1/q_0 = 2^8$, 2^9 and 2^{10} . The sliding velocity v is set to 0.1 m/s. From (a) to (d) $q_0 d 10^3 = 0.63, 3.2, 4.4, 6.3$.

stiffening, as instead was the case of Fig. 7(a). Finally, for intermediate coating thicknesses we find, as expected, an intermediate scenario for both friction and contact area, see Figs. 6(b), 6(c), 7(b) and 7(c).

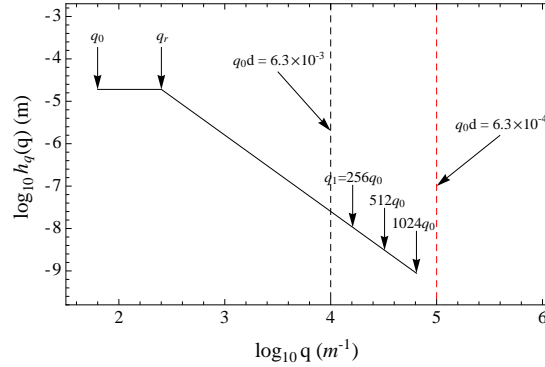


Figure 8. Wavelength representative amplitude $h_q(q) \approx q_0 \sqrt{C(q)}$ as a function of the wavenumber q , in \log_{10} - \log_{10} scale, corresponding to the PSD of Fig. 5. The vertical dashed lines indicate the roughness frequencies q corresponding to $\bar{q} = qd = 1$ for the thicker (black line, with $q_0 d = 6.3 \cdot 10^{-3}$) and thinner (red, with $q_0 d = 6.3 \cdot 10^{-4}$) coatings adopted in the simulations, where d is the coating thickness.

In Fig. 8 we show the wavelength representative amplitude $h_q(q) \approx q_0 \sqrt{C(q)}$ as a function of the wavenumber q , in \log_{10} - \log_{10} scale, corresponding to the PSD of Fig. 5. The vertical dashed lines indicate the roughness frequencies q corresponding to $\bar{q} = qd = 1$ for the thicker (black line, with $q_0 d = 6.3 \cdot 10^{-3}$) and thinner (red, with $q_0 d = 6.3 \cdot 10^{-4}$) coatings adopted in the simulations. From the theory developed in Appendix A, the layer thickness enters the theory mainly through $\bar{q} = \tanh(\bar{q})$, where again $\bar{q} = qd$. Thus we can qualitatively observe that for frequencies $\bar{q} \geq \bar{q}_{\text{high}} = 2$,

$\bar{q} \approx 1$ so that the roughness wavelengths approximately smaller than the coating thickness are unable to probe the sub-coating composite rheology, whereas for $\bar{q} \leq \bar{q}_{\text{low}} = 0.1$, $\bar{q} \approx \bar{q}$ so that the roughness asperities do mainly probe the bulk. However, whilst we expect the exact values of \bar{q}_{high} (and \bar{q}_{low}) to be quantitatively affected not only by the geometrical composite characteristics, but also from its rheological properties (see e.g. the discussion in Sec. V), it is worth in the present context (where both coating and bulk show a Young's modulus of similar order of magnitude) to show \bar{q}_{high} (and \bar{q}_{low}) in Fig. 8 for both the thinner [Fig. 9(a)] and thicker [Fig. 9(b)] coating.

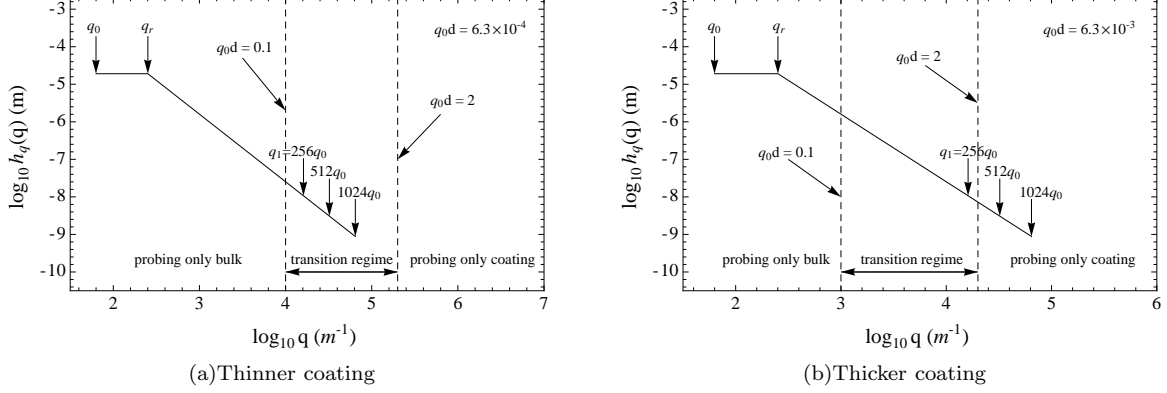


Figure 9. Wavelength representative amplitude $h_q(q) \approx q_0 \sqrt{C'(q)}$ as a function of the wavenumber q , in \log_{10} - \log_{10} scale, corresponding to the PSD of Fig. 5. The vertical dashed lines indicate the roughness frequencies q corresponding to $\bar{q} = 0.1$ and 2. (a) for the thinner ($q_0 d = 6.3 \cdot 10^{-4}$), and (b) thicker ($q_0 d = 6.3 \cdot 10^{-3}$) coating.

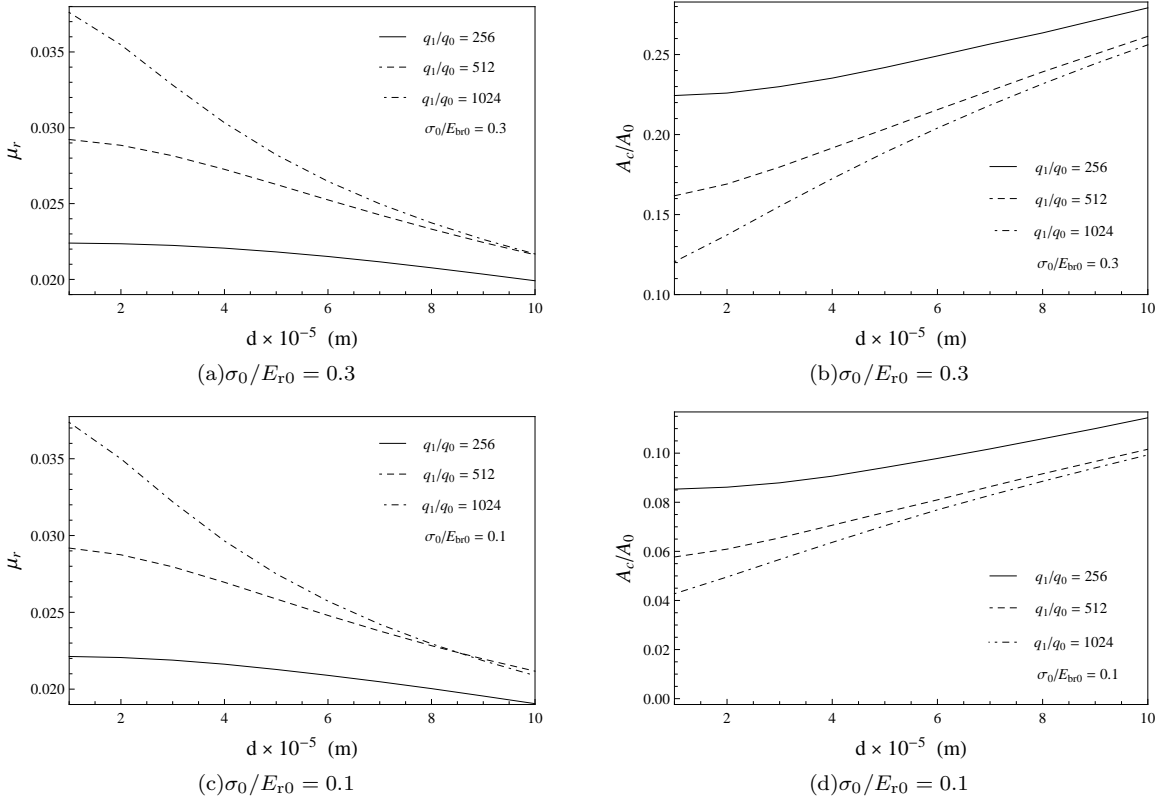


Figure 10. (a) Micro-rolling friction and (b) normalized projected contact area as a function of the coating thickness, for $q_1/q_0 = 2^8$, 2^9 and 2^{10} . The sliding velocity v is set to 0.1 m/s.

We observe in Fig. 9(a) (thinner coating) that, accordingly to the previous arguments, the whole range of roughness frequencies can probe the bulk, whereas for the thicker coating [Fig. 9(b)] the smallest wavelengths are not aware of

the bulk, confirming the friction and contact area results reported in Figs. 6 and 7.

Finally, in Fig. 10 we show (a,c) the micro-rolling friction μ_r and (b,d) the contact area A_c/A_0 as a function of the coating thickness, for $q_1/q_0 = 2^8$, 2^9 and 2^{10} (sliding velocity v set to 0.1 m/s), and for a contact pressure $\sigma_0/E_{r0} = 0.1$ and 0.3 (qualitatively similar behaviours characterize the interaction at different contact pressures, thus not shown here for the sake of brevity). It is interesting to observe that the friction (and contact area) curve for $q_1/q_0 = 2^{10}$ converges to the $q_1/q_0 = 2^9$ curve at increasing values of coating thickness, again as expected from the previous arguments. A similar conclusion applies for the contact area, see Figs. 10(b) and 10(d)⁴. Thus, roughness frequencies larger than $q_1/q_0 = 2^9$ do not affect the interfacial contact mechanics at such coating size, supporting the general statement that a physically meaningful characterization (and prediction) of the friction and contact area properties of a generic interaction can only be obtained provided that both confinement rheology and surface physics, as well as surface roughness, are fully characterized to a same degree of completeness.

V. DISCUSSION

The multiscale nature of the hysteretic friction μ_r and contact area A_c/A_0 for randomly rough interactions is nowadays well accepted among contact mechanics researchers, mainly thanks to the theoretical achievements of the Persson [2]. At a contact scale of representative size $\lambda = 2\pi/q$, where $\zeta = q/q_0$ is the magnification at which the contact is observed with respect to a contact macroscale $L_0 = 2\pi/q_0$, the dissipation is confined in a bulk volume λ^3 and corresponds approximately to a friction force $F_T \approx \lambda^3 q \operatorname{Im}[E(\omega)] h_\lambda^2 / \lambda^2$. Here $\omega = \mathbf{q} \cdot \mathbf{v}$ is the angular frequency of excitation with \mathbf{v} as the sliding velocity, $E(\omega)$ is the complex Young's modulus of rubber, and h_λ is the amplitude of the roughness wavelength λ . Thus, the frictional shear stress is $\tau_T \approx q^2 \operatorname{Im}[E(\omega)] h_\lambda^2$. Hence, for a rough surface with self affine characteristics, one has that the contribution to friction $\Delta\tau_T$ related to the roughness wavelength λ is

$$\Delta\tau_T / \Delta q \approx \operatorname{Im}[E(\omega)] q^3 C(q) \propto \operatorname{Im}[E(\mathbf{q} \cdot \mathbf{v})] q^{-2H+1}, \quad (15)$$

where

$$C(q) = (2\pi)^{-2} \int d^2\mathbf{x} \langle h(\mathbf{x}) h(\mathbf{0}) \rangle e^{-i\mathbf{q} \cdot \mathbf{x}}$$

is the power spectral density of the surface roughness, $h(\mathbf{x})$ is the substrate height measured from the average surface plane, and H is the Hurst coefficient. (15) does not show any cut-off mechanism of friction, and moreover for fractal dimensions $D_f > 2.5$ the contribution to dissipation generated by decreasing roughness length scales is even unbounded for ideal (infinite) systems. Similar considerations apply for the contact area. In particular, by observing that $A(\zeta) \bar{\sigma}(\zeta) = \sigma_0 A_0 = F_N$, where $A(\zeta)$ is the contact area obtained when an arbitrary high-frequency cut-off is applied to the PSD (i.e. $C(q > \zeta q_0) = 0$), one simply has

$$\frac{dA(\zeta)}{d\zeta} \propto -\frac{d\bar{\sigma}(\zeta)}{d\zeta}.$$

By approximating $d\bar{\sigma}(\zeta)/d\zeta$ with $\sqrt{d\langle\sigma^2\rangle/d\zeta}$, where

$$d\langle\sigma^2\rangle = \frac{E_{r0}^2}{4} d[m_{2,\text{eff}}(\zeta)]$$

with

$$m_{2,\text{eff}}(\zeta) = \int_{q_0}^{\zeta q_0} dq^2 q^2 C(q) |E_{r,\theta}(\mathbf{q} \cdot \mathbf{v})|^2 / E_{r0}^2$$

$$d[m_{2,\text{eff}}(\zeta)] \approx d\zeta q_0 q^3 C(q) |E_{r,\theta}(\mathbf{q} \cdot \mathbf{v})|^2 / E_{r0}^2$$

and where $|E_{r,\theta}(\mathbf{q} \cdot \mathbf{v})| = (2\pi)^{-1} \int d\theta |E_{r,\theta}(qv \cos \theta)|$ (assuming $v_y = 0$). Thus, approximately,

$$\Delta A(\zeta) / \Delta q \propto -|E_{r,\theta}(\mathbf{q} \cdot \mathbf{v}_0)| q^{(-2H+1)/2}. \quad (16)$$

⁴The contact area increases at larger coating thickness d since, for the adopted composite, by increasing d a wider range of PSD wavelengths is allowed to probe only the coating, which is not subjected to a sliding-induced viscoelastic stiffening. Hence, an increasing amount of roughness wavelengths is in full-contact with the substrate.

We first observe that $\Delta A(\zeta)/\Delta q < 0$, i.e. the contact continuously decreases by increasing the small scale roughness spectral content. The power law exponent is similar to the previous case (thus, similar considerations apply here), and no cut-off mechanism of contact area appears. Thus, accordingly to both (15) and 16, a small-scale cut-off mechanism can only be introduced through the effective rheological response of the composite $E(\omega)$. Whilst this has been already numerically proved in Sec. IV for the case of an elastic coating bonded onto a rubber bulk, however, we will show (a more interesting feature) in the following that the multiscale design of the effective complex modulus $E(\omega)$ (i.e. the choice of the composite materials arrangement) can be adopted to provide extremely tailored contact mechanics properties, such as (but not limited to) an enhanced micro-rolling friction.

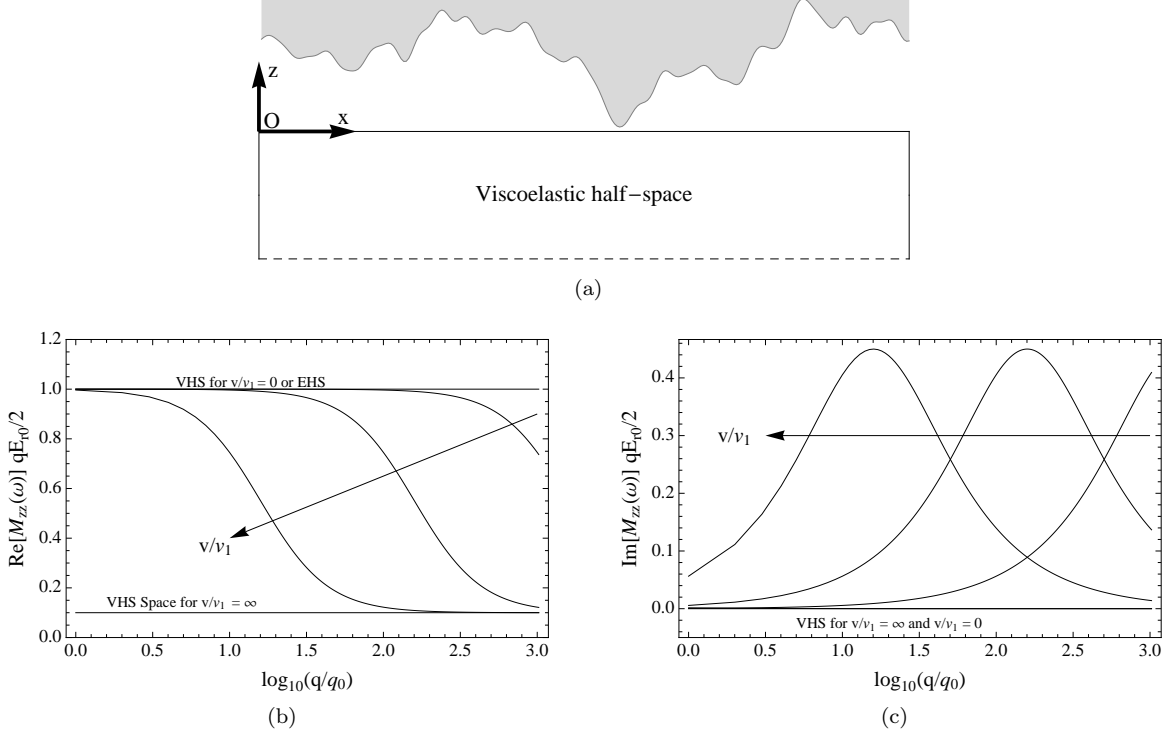


Figure 11. (a) Schematic of a viscoelastic half space (VHS) in sliding contact with a rigid rough surface. (b) Real and (c) imaginary part of the dimensionless surface response $M_{zz}(\omega) q E_{r0}/2$ (with $\omega = qv$ and $q_y = 0$) as a function of the wave number q/q_0 ($q_0 = 2\pi/L_0$). The bulk is characterized by a single relaxation time $\tau = L_0/v_1$ and $E_{r\infty}/E_{r0} = 10$ (with $\nu(\omega) = \nu_0$). For the dimensionless sliding velocities v/v_1 in the set $[0, 10^{-4}, 10^{-3}, 10^{-2}, \infty]$. EHS is for elastic half space.

In Fig.11 we show, for a viscoelastic half space in sliding contact with a generic rigid rough surface, the cross section (at $q_y = 0$) of the (b) real and (c) imaginary part of the dimensionless surface response $M_{zz}(\omega) / [2 / (q E_{r0})]$ (with $\omega = qv$) as a function of the wave number q/q_0 (with $q_0 = 2\pi/L_0$). The bulk is characterized by a single relaxation time $\tau = L_0/v_1$ and by $E_{r\infty}/E_{r0} = 10$. Several dimensionless sliding velocities v/v_1 are reported, belonging to the set $[0, 10^{-4}, 10^{-3}, 10^{-2}, \infty]$. For $v/v_1 \rightarrow 0$ (∞), the solid is elastically probed in its rubbery or relaxed (glassy) regime, see also Fig. 11(b). For intermediate sliding velocities, the roughness wavelengths probe the rubber at different degree of stiffening, and in particular a monotonic rubber stiffening occurs at increasing roughness frequencies [Fig. 11(b)]. Fig. 11(c) shows the imaginary part corresponding to Fig. 11(b). We observe as expected that, by varying the sliding speed, a different range of roughness wavelengths can probe the rubber at the highest dissipation. However, no cut-off mechanism occurs in both Figs. 11(b) and 11(c), as previously discussed.

In Fig. 12 the case of a viscoelastic half space coated with an elastic layer in sliding contact with a rigid rough surface is reported. In particular, we show the cross section (at $q_y = 0$) of the (b) real part of the dimensionless surface response $M_{zz}(\omega) / [2 / (q E_{r0})]$ and (c) effective composite loss tangent $\text{Im}[M_{zz}(\omega)^{-1}] / \text{Re}[M_{zz}(\omega)^{-1}]$ (with $\omega = qv$) as a function of the wave number q/q_0 (with $q_0 = 2\pi/L_0$). The bulk is characterized by a single relaxation time $\tau = L_0/v_1$ and $E_{r\infty}/E_{r0} = 10$, whereas the coating has a reduced elastic modulus E_{r0} . The dimensionless coating thickness $q_0 d$ is let to vary in the set $[0, 9.4, 31, 63, 94, 190, \infty] 10^{-3}$, with $v/v_1 = 0.02$ (unless differently specified). In Fig. 12(b), the red curve corresponds to the limiting case of an elastic coating with a negligible thickness; thus, the composite undergoes a monotonic stiffening at increasing roughness frequencies, as previously reported in Fig.

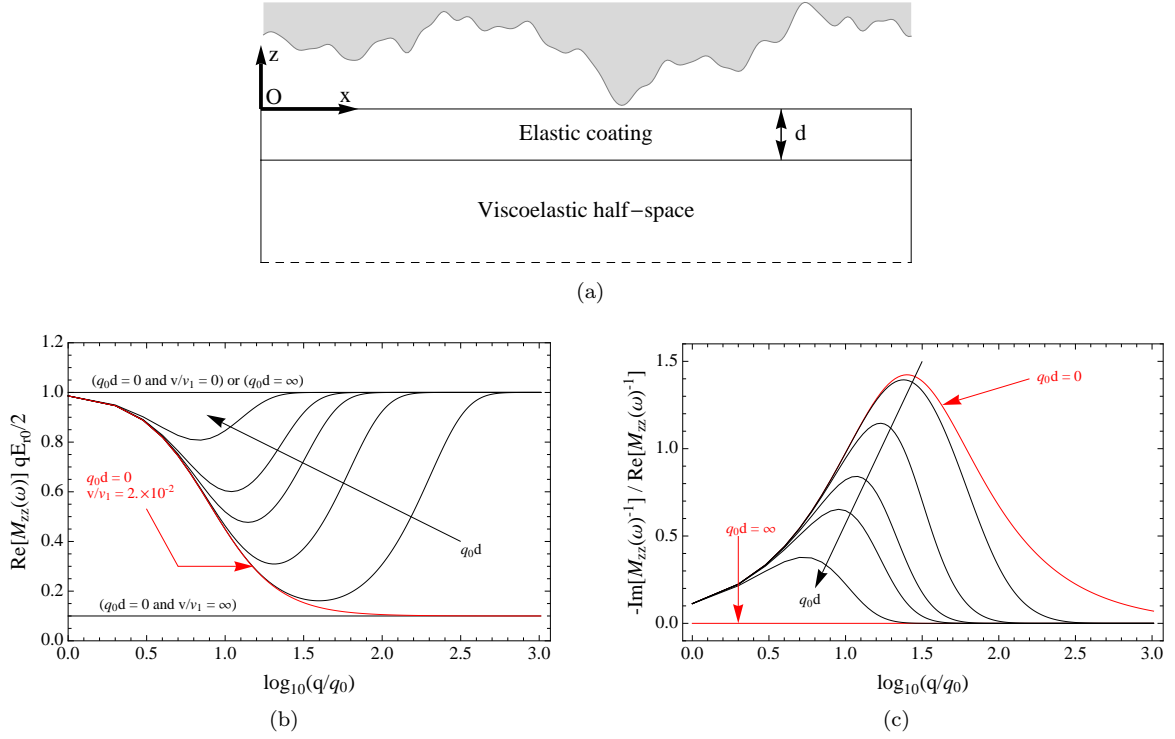


Figure 12. (a) Schematic of a viscoelastic half space coated with an elastic layer in sliding contact with a rigid rough surface. (b) Real part of the dimensionless surface response $M_{zz}(\omega) q E_{r0}/2$ and (c) effective composite loss tangent $\text{Im}[M_{zz}(\omega)^{-1}] / \text{Re}[M_{zz}(\omega)^{-1}]$ (with $\omega = qv$ and $q_y = 0$) as a function of the wave number q/q_0 ($q_0 = 2\pi/L_0$). The bulk is characterized by a single relaxation time $\tau = L_0/v_1$ and $E_{r\infty}/E_{r0} = 10$ (with constant Poisson ratio). The coating has a reduced elastic modulus E_{r0} . For the dimensionless coating thickness $q_0 d$ in the set $[0, 9.4, 31, 63, 94, 190, \infty] 10^{-3}$, with $v/v_1 = 0.02$ (unless differently specified).

11(b). For increasing coating size $q_0 d$, interestingly, the effective surface response shows a minimum for intermediate frequencies, whereas both large and small wavelengths probe the composite in its compliant regime (corresponding to the rubber relaxed modulus E_{r0}). Thus, for such a composite, increasing the small-scale roughness content is expected not to affect the true contact area, as previously demonstrated with the arguments of Sec. IV. In term of effective loss tangent, Fig. 12(c) shows as red curves the two limiting cases of coating with negligible thickness ($q_0 d = 0$) and with infinite thickness ($q_0 d \rightarrow \infty$). For $q_0 d \rightarrow \infty$ the contact regime occurs under pure elasticity, obviously determining a null dissipation and, correspondingly, a null loss tangent. For intermediate coating sizes, the loss tangent loses the classical bell shape (in the log scale) and, interestingly, a dissipation cut-off frequency q_μ appears, so that wavelengths smaller than $\approx q_\mu^{-1}$ do not quantitatively probe the viscoelastic bulk, with no resulting contribution in term of hysteretic friction (even considering that those small scale wavelengths are in full contact with the composite). Thus, this behaviour generates the physical scenario presented in Sec. IV.

Finally, Fig. 13 shows the case of (a) a composite block constituted by a rubber bulk coated by three rubber layers (with different rheologies) in sliding contact with a rigid rough surface. In particular, we show the cross section (at $q_y = 0$) of the (b) effective composite loss tangent $\text{Im}[M_{zz}(\omega)^{-1}] / \text{Re}[M_{zz}(\omega)^{-1}]$ (with $\omega = qv$ and $q_y = 0$) as a function of the wave number q/q_0 ($q_0 = 2\pi/L_0$). The bulk is characterized by a single relaxation time $\tau = L_0/v_1$ and $E_{r\infty}/E_{r0} = 10$, whereas the coatings are viscoelastic layers with the same rubbery (E_{r0}) and glassy ($E_{r\infty}$) modulus of the bulk, but with different relaxation time τ_j , where $j = 1, 2, 3$ (1 for the innermost layer). In particular, $\tau_j q_0 v_1 = 2s_j \pi$, with $s_j = [10^{-1}, 10^{-2}, 10^{-3}]$, whereas the coating size follows the rule $q_0 d_j = 2s_j \pi$. The sliding speed is set to $v/v_1 = 1$. In Fig. 13(b) the dashed curves represent the loss tangent of each composite layer, as shown in descriptive balloons, whereas the continuous line is the effective loss tangent. We observe that the properly-designed building of the composite layers allows to provide effective block dissipation characteristics which are almost constant and independent from the probing roughness wavelengths, e.g. in order to let the viscoelastic friction be maximized. More generally, however, our model can be adopted to determine the optimal composite packaging which provides tailored contact properties, such as friction or adherence. Furthermore, the interface can be designed to be roughness specific, i.e. providing contact mechanics properties only over a windowed roughness spectral content, e.g.

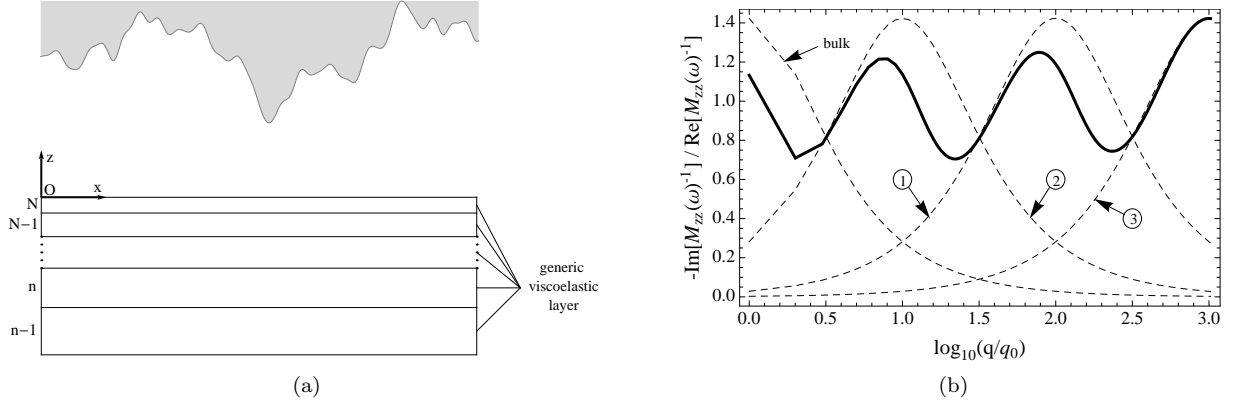


Figure 13. (a) Schematic of a composite block constituted by a rubber bulk coated by three rubber layers (with different rheologies) in sliding contact with a rigid rough surface. (b) Effective composite loss tangent $\text{Im}[M_{zz}(\omega)^{-1}] / \text{Re}[M_{zz}(\omega)^{-1}]$ (with $\omega = qv$ and $q_y = 0$) as a function of the wave number q/q_0 ($q_0 = 2\pi/L_0$). The bulk is characterized by a single relaxation time $\tau = L_0/v_1$ and $E_{r\infty}/E_{r0} = 10$. The coatings are viscoelastic layers with the same rubbery (E_{r0}) and glassy ($E_{r\infty}$) modulus of the bulk, but with different relaxation time τ_j , where $j = 1$ for the innermost layer. In particular, $\tau_j q_0 v_1 = 2s_j\pi$, with $s_j = [10^{-1}, 10^{-2}, 10^{-3}]$, whereas the coating size follows the rule $q_0 d_j = 2s_j\pi$. For sliding velocities $v/v_1 = 1$. In (b) the dashed curves represent the loss tangent of each composite layer, as shown in descriptive balloons (corresponding to s_j), whereas the continuous line is the effective loss tangent.

for biological sensing, bio-adhesion, tire grip control, to cite some.

VI. CONCLUSIONS

We have presented the first numerical contact mechanics model for (randomly or deterministic) rough surfaces, to be applied for the prediction of the rough contact mechanics of a general viscoelastic block, with graded rheology, in steady sliding contact with a rough rigid surface. In particular, our model is able to handle both stepwise or continuously-graded block rheologies, with a (reduced) computational effort typical of the residuals molecular dynamics scheme. We have critically discussed on the role of small-scale wavelengths on rubber friction and contact area, and we showed for the first time that the rough contact mechanics exhibits effective interface properties which converge to asymptotes upon increase of the small-scale roughness content, under the adoption of some realistic description of the rheology of the confinement. Furthermore, we show that our model can be effectively adopted for the design of the composite-layers packaging providing contact mechanics characteristics (such as friction and adhesion) tailored to be roughness specific, e.g. for biological sensing, bio-adhesion, tire grip control, to cite some possible applications.

ACKNOWLEDGMENTS

DC gratefully acknowledges the support the European Research Council (ERC starting researcher grant 'INTER-FACES', No. 279439).

Appendix A: General theory for the finite thickness slab with homogeneous rheological properties

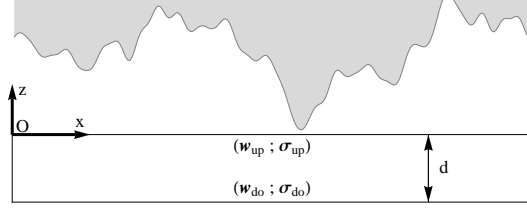


Figure 14. Schematic of an infinitely-wide layer with finite thickness d , with generic linearly-viscoelastic rheological properties.

In this section we solve the Navier's equation for isotropic viscoelasticity in the case of a finite-thickness infinitely wide slab. Whilst we make use of the Persson's complex solution presented in [2], in this section the theory will be further developed to a more general case of frequency-varying Poisson's ratio $\nu(\omega)$. This newly-developed formulation will then be applied to determine the effective surface response $M_{zz}(q, \omega)$ of a composite with, respectively, stepwise rheology in Sec. III and continuously-graded rheology in Appendix B.

In particular, we consider the case of an infinitely-wide homogeneous slab of height d , with isotropic linearly-viscoelastic rheological properties. Furthermore, we assume the contact to occur under isothermal conditions, and the rough surface height $h(\mathbf{x})$ [with $\langle h(\mathbf{x}) \rangle = (L_x L_y)^{-1} \int d^2x h(\mathbf{x}) = 0$] to be characterized by a mean square surface slope $m_2 \ll 1$, where $m_2 = \langle |\nabla h|^2 \rangle$. The Navier's equation for a viscoelastic medium reads

$$\rho \frac{\partial^2 \mathbf{w}}{\partial t^2} = \mu \nabla^2 \mathbf{w} + (\mu + \lambda) \nabla \nabla \cdot \mathbf{w}, \quad (\text{A1})$$

where μ and λ are the complex viscoelastic Lamé' parameters, ρ the density and $w(\mathbf{x}, z)$ the displacement field. We define the following Fourier transforms

$$\begin{aligned} \mathbf{w}(\mathbf{q}, z, \omega) &= (2\pi)^{-3} \int dt \int d^2x \mathbf{w}(\mathbf{x}, z, t) e^{-i(\mathbf{q} \cdot \mathbf{x} - \omega t)} \\ \mu(\omega) &= \int dt \mu(t) e^{-i(-\omega t)} \end{aligned}$$

and, inversely,

$$\begin{aligned} \mu(t) &= (2\pi)^{-1} \int d\omega e^{i(-\omega t)} \mu(\omega) \\ \mathbf{w}(\mathbf{x}, z, t) &= \int dt \int d^2x \mathbf{w}(\mathbf{q}, \omega) e^{i(\mathbf{q} \cdot \mathbf{x} - \omega t)}. \end{aligned}$$

Thus, by making use of the field decomposition suggested by Persson [2] (let \mathbf{e}_3 be a unit vector along the z axis, see Fig. 2) and by defining $\bar{\nabla} = -i\nabla = (\mathbf{q} - \mathbf{e}_3 i \frac{\partial}{\partial z})$, the Fourier transform ($\mathbf{x} \rightarrow \mathbf{q}$ and $t \rightarrow \omega$) of (A1) results in

$$\left(\omega^2 \rho - \mu q^2 + \mu \frac{\partial^2}{\partial z^2} \right) \mathbf{w} = (\mu + \lambda) \bar{\nabla} [\bar{\nabla} \cdot \mathbf{w}].$$

Note that $\bar{\nabla}^2 = q^2 - \frac{\partial^2}{\partial z^2}$. Accordingly, we define $\mathbf{p} = \mathbf{e}_3 \times \bar{\nabla} = \mathbf{e}_3 \times \mathbf{q}$, and we decompose the displacement field $\mathbf{w}(\mathbf{q}, z, \omega)$ into [2]

$$\mathbf{w} = \bar{\nabla} A(\mathbf{q}, z, \omega) + \mathbf{p} B(\mathbf{q}, z, \omega) + \bar{\nabla} \times \mathbf{p} C(\mathbf{q}, z, \omega), \quad (\text{A2})$$

resulting in

$$(\omega^2 \rho - \mu \bar{\nabla}^2) \mathbf{w} = (\mu + \lambda) \bar{\nabla} [\bar{\nabla}^2 A(\mathbf{q}, z, \omega)],$$

where

$$\begin{aligned} w_1 &= (q_x A - q_y B + i q_x C') \\ w_2 &= (q_y A + q_x B + i q_y C') \\ w_3 &= (q^2 C - i A'). \end{aligned}$$

This results in three independent equations in the three scalar fields

$$\begin{aligned} \left[\omega^2 - \frac{2\mu + \lambda}{\rho} \bar{\nabla}^2 \right] A(\mathbf{q}, z, \omega) &= 0 \\ \left[\omega^2 - \frac{\mu}{\rho} \bar{\nabla}^2 \right] B(\mathbf{q}, z, \omega) &= 0 \\ \left[\omega^2 - \frac{\mu}{\rho} \bar{\nabla}^2 \right] C(\mathbf{q}, z, \omega) &= 0, \end{aligned}$$

and by defining $c_T^2 = \mu/\rho$ and $c_L^2 = (2\mu + \lambda)/\rho$ (where $\frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}$, $2 + \frac{\lambda}{\mu} = \frac{2-2\nu}{1-2\nu} = c_L^2/c_T^2$)

$$\begin{aligned} \left[\omega^2/c_L^2 - q^2 + \frac{\partial^2}{\partial z^2} \right] A(\mathbf{q}, z, \omega) &= 0 \\ \left[\omega^2/c_T^2 - q^2 + \frac{\partial^2}{\partial z^2} \right] B(\mathbf{q}, z, \omega) &= 0 \\ \left[\omega^2/c_T^2 - q^2 + \frac{\partial^2}{\partial z^2} \right] C(\mathbf{q}, z, \omega) &= 0. \end{aligned}$$

Given a solution $A(\mathbf{q}, z, \omega) = A(\mathbf{q}, \omega) e^{f_L z}$ [and $B(\mathbf{q}, z, \omega) = B(\mathbf{q}, \omega) e^{f_T z}$, $C(\mathbf{q}, z, \omega) = C(\mathbf{q}, \omega) e^{f_T z}$], we get that $f = \pm q \sqrt{1 - \alpha^2 + i\varepsilon}$, where ε is a small positive number (branch cut along π) and $\alpha^2 = \omega^2/(q^2 c^2)$ (with $\alpha_L^2/\alpha_T^2 = \frac{1-2\nu(\omega)}{2-2\nu(\omega)}$) [2]. Hence

$$\begin{aligned} A(\mathbf{q}, z, \omega) &= [A_1(\mathbf{q}, \omega) e^{f_L z} + A_2(\mathbf{q}, \omega) e^{-f_L z}] \\ B(\mathbf{q}, z, \omega) &= [B_1(\mathbf{q}, \omega) e^{f_T z} + B_2(\mathbf{q}, \omega) e^{-f_T z}] \\ C(\mathbf{q}, z, \omega) &= [C_1(\mathbf{q}, \omega) e^{f_T z} + C_2(\mathbf{q}, \omega) e^{-f_T z}]. \end{aligned} \tag{A3}$$

Moreover, the constitutive relationship can be applied

$$\mathbf{S}(\mathbf{q}, z, \omega) = \mu (\nabla \mathbf{w} + \mathbf{w} \nabla) + i \lambda \bar{\nabla}^2 A \mathbf{I},$$

where \mathbf{S} is the stress tensor, which along the z -direction [where $(\nabla \mathbf{w} + \mathbf{w} \nabla) \mathbf{e}_3 = w_{3,l} + w_{l,3}$] reads

$$\sigma_l(\mathbf{q}, z, \omega) = i\mu \left(\bar{\nabla}_l w_3 - i \frac{\partial w_l}{\partial z} \right) + i \lambda \bar{\nabla}^2 A \delta_{l3}. \tag{A4}$$

The index $l = 1, 2, 3$ is used here in substitution of the reference coordinate x, y , and z respectively. From (A4) it results

$$\begin{aligned} \sigma_1(\mathbf{q}, z, \omega) &= \mu [2q_x A' - q_y B' + i q_x (q^2 + f_T^2) C] \\ \sigma_2(\mathbf{q}, z, \omega) &= \mu [2q_y A' + q_x B' + i q_y (q^2 + f_T^2) C] \\ \sigma_3(\mathbf{q}, z, \omega) &= i [\lambda q^2 - (2\mu + \lambda) f_L^2] A + 2\mu q^2 C'. \end{aligned}$$

To determine the six-scalar fields of (A3), we apply the following boundary conditions to the lower side of the slab (see Fig. 14)

$$\begin{aligned} \sigma_l(\mathbf{q}, -d, \omega) &= \sigma_{l,\text{do}}(\mathbf{q}, \omega) \\ u_l(\mathbf{q}, -d, \omega) &= u_{l,\text{do}}(\mathbf{q}, \omega). \end{aligned}$$

Hence, the stress and displacement fields on the upper surface ($z = 0$) can be easily determined, and in particular in

the quasistatic regime ($\alpha \ll 1$, i.e. $\omega/(qc) = v/c \ll 1$) $\boldsymbol{\sigma}_{\text{up}} = \boldsymbol{\sigma}(\mathbf{q}, z=0, \omega)$ and $\mathbf{w}_{\text{up}} = \mathbf{w}(\mathbf{q}, z=0, \omega)$ read

$$\begin{aligned}\boldsymbol{\sigma}_{\text{up}}/[E_r(\omega)q/2] &= \cosh(qd) [\mathbf{M}_1 \boldsymbol{\sigma}_{\text{low}}/[E_r(\omega)q/2] + \mathbf{M}_2 \mathbf{w}_{\text{low}}] \\ \mathbf{w}_{\text{up}} &= \cosh(qd) [\mathbf{M}_3 \boldsymbol{\sigma}_{\text{low}}/[E_r(\omega)q/2] + \mathbf{M}_4 \mathbf{w}_{\text{low}}],\end{aligned}$$

i.e. in matrix form

$$\begin{bmatrix} \boldsymbol{\sigma}_{\text{up}}/[E_r(\omega)q/2] \\ \mathbf{w}_{\text{up}} \end{bmatrix} = \cosh(qd) \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_{\text{low}}/[E_r(\omega)q/2] \\ \mathbf{w}_{\text{low}} \end{bmatrix},$$

where $E_r(\omega) = E(\omega)/[1 - \nu(\omega)^2]$ is the complex reduced elastic modulus. $\bar{q} = qd$ (and similarly for q_x and q_y), $\tilde{q} = \tanh \bar{q}$, $p = 1 - \nu(\omega)$, $p_0 = 1 - \nu_0$ [with $\nu_0 = \nu(\omega \rightarrow 0)$] and we have defined

$$\begin{aligned}m &= p/p_0, \quad n = [1 - 2\nu(\omega)]/[1 - 2\nu_0], \quad \gamma = n/m, \\ \beta &= \frac{1 - 4\nu p_0}{[1 - 2\nu(\omega)][1 - 2\nu_0]}.\end{aligned}\tag{A5}$$

Note that m , n , γ and β depends on the frequency ω through the dependence on $\nu(\omega)$. After simplifications we obtain

$$M_1 = I + (2\bar{q}p_0)^{-1} \begin{bmatrix} \bar{q}_x^2 \tilde{q} & \bar{q}_x \bar{q}_y \tilde{q} & -i\bar{q}_x \gamma [q - (1 - 2\nu_0)\tilde{q}] \\ \bar{q}_x \bar{q}_y \tilde{q} & \bar{q}_y^2 \tilde{q} & -i\bar{q}_y \gamma [q - (1 - 2\nu_0)\tilde{q}] \\ -i\bar{q}_x [\bar{q} + \beta(1 - 2\nu_0)\tilde{q}] & -i\bar{q}_y [\bar{q} + \beta(1 - 2\nu_0)\tilde{q}] & -\gamma \bar{q}^2 \tilde{q} \end{bmatrix},\tag{A6}$$

$$M_2 = \bar{q}^{-2} \begin{bmatrix} (n\bar{q} + \tilde{q}) \bar{q}_x^2 + p\tilde{q}\bar{q}_y^2 & \bar{q}_x \bar{q}_y (n\bar{q} + \nu\tilde{q}) & -im\bar{q}_x \bar{q}^2 \tilde{q} \\ \bar{q}_x \bar{q}_y (n\bar{q} + \nu\tilde{q}) & p\tilde{q}\bar{q}_x^2 + (n\bar{q} + \tilde{q}) \bar{q}_y^2 & -im\bar{q}_y \bar{q}^2 \tilde{q} \\ -in\bar{q}_x \bar{q}^2 \tilde{q} & -in\bar{q}_y \bar{q}^2 \tilde{q} & -m\bar{q}^2 (\bar{q} - \tilde{q}/n) \end{bmatrix},\tag{A7}$$

$$M_3 = (2\bar{q}mp_0)^{-2} \begin{bmatrix} 4p\tilde{q}\bar{q}^2 + m(\bar{q} - \tilde{q}) \bar{q}_x^2 & m\bar{q}_x \bar{q}_y (\bar{q} - \tilde{q}) & -in\bar{q}_x \bar{q}^2 \tilde{q} \\ m\bar{q}_x \bar{q}_y (\bar{q} - \tilde{q}) & 4p\tilde{q}\bar{q}^2 + m(\bar{q} - \tilde{q}) \bar{q}_y^2 & -in\bar{q}_y \bar{q}^2 \tilde{q} \\ -im\bar{q}_x \bar{q}^2 \tilde{q} & -im\bar{q}_y \bar{q}^2 \tilde{q} & -\gamma \bar{q}^2 [m(\bar{q} + \tilde{q}) - 4p\tilde{q}] \end{bmatrix}\tag{A8}$$

and

$$M_4 = I + (2\bar{q}p)^{-1} \begin{bmatrix} \gamma \bar{q}_x^2 \tilde{q} & \gamma \bar{q}_x \bar{q}_y \tilde{q} & -i\bar{q}_x [q + (1 - 2\nu_0)\tilde{q}] \\ \gamma \bar{q}_x \bar{q}_y \tilde{q} & \gamma \bar{q}_y^2 \tilde{q} & -i\bar{q}_y [q + (1 - 2\nu_0)\tilde{q}] \\ -i\bar{q}_x \gamma [q - \beta(1 - 2\nu_0)\tilde{q}] & -i\bar{q}_y \gamma [q - \beta(1 - 2\nu_0)\tilde{q}] & -\bar{q}^2 \tilde{q} \end{bmatrix}.\tag{A9}$$

Observe that $M_j = M_j(\omega)$ through the frequency dependence of the Poisson's ratio.

In the case the frequency variation of the lateral contraction can be neglected, i.e. $\nu(\omega) = \nu = \nu_0$, we have that $m = n = \beta = \gamma = 1$ (and $p_0 = p$) and the (A6)-(A9) simplify to

$$\begin{aligned}M_1 &= I + (2\bar{q}p)^{-1} \begin{bmatrix} \bar{q}_x^2 \tilde{q} & \bar{q}_x \bar{q}_y \tilde{q} & -i\bar{q}_x [\bar{q} - (2p - 1)\tilde{q}] \\ \bar{q}_x \bar{q}_y \tilde{q} & \bar{q}_y^2 \tilde{q} & -i\bar{q}_y [\bar{q} - (2p - 1)\tilde{q}] \\ -i\bar{q}_x [\bar{q} + (2p - 1)\tilde{q}] & -i\bar{q}_y [\bar{q} + (2p - 1)\tilde{q}] & -\bar{q}^2 \tilde{q} \end{bmatrix} \\ M_2 &= \bar{q}^{-2} \begin{bmatrix} (\bar{q} + \tilde{q}) \bar{q}_x^2 + p\tilde{q}\bar{q}_y^2 & \bar{q}_x \bar{q}_y (\bar{q} + \nu\tilde{q}) & -i\bar{q}_x \bar{q}^2 \tilde{q} \\ \bar{q}_x \bar{q}_y (\bar{q} + \nu\tilde{q}) & p\tilde{q}\bar{q}_x^2 + (\bar{q} + \tilde{q}) \bar{q}_y^2 & -i\bar{q}_y \bar{q}^2 \tilde{q} \\ -i\bar{q}_x \bar{q}^2 \tilde{q} & -i\bar{q}_y \bar{q}^2 \tilde{q} & -\bar{q}^2 (\bar{q} - \tilde{q}) \end{bmatrix} \\ M_3 &= (2\bar{q}p)^{-2} \begin{bmatrix} 4p\tilde{q}\bar{q}^2 + (\bar{q} - \tilde{q}) \bar{q}_x^2 & \bar{q}_x \bar{q}_y (\bar{q} - \tilde{q}) & -i\bar{q}_x \bar{q}^2 \tilde{q} \\ \bar{q}_x \bar{q}_y (\bar{q} - \tilde{q}) & 4p\tilde{q}\bar{q}^2 + (\bar{q} - \tilde{q}) \bar{q}_y^2 & -i\bar{q}_y \bar{q}^2 \tilde{q} \\ -i\bar{q}_x \bar{q}^2 \tilde{q} & -i\bar{q}_y \bar{q}^2 \tilde{q} & -\bar{q}^2 [(\bar{q} + \tilde{q}) - 4p\tilde{q}] \end{bmatrix},\end{aligned}$$

with $M_1 = M_4^T$.

1. Limiting cases for the single slab

For the **slab constrained onto a rigid substrate** [given by $\mathbf{M}_3\mathbf{M}_1^{-1}$, see (8)] we have after simplifications

$$\frac{M_{zz}(\mathbf{q}, \omega)}{E_r(\omega) q/2} = m^{-1} \frac{(4p_0 - 1) \tilde{q} - \bar{q} (1 - \tilde{q}^2)}{4p_0^2/\gamma - \bar{q}\tilde{q} [(1 - 2p_0)(\beta - 1) + 2p_0(\gamma - 1)/\gamma] + \bar{q}^2 (1 - \tilde{q}^2) - \beta\tilde{q}^2 (1 - 2p_0)^2}, \quad (\text{A10})$$

whereas for the case of **free-standing slab** [given by $\mathbf{M}_4\mathbf{M}_2^{-1}$, see (9)] we have

$$\frac{M_{zz}(\mathbf{q}, \omega)}{E_r(\omega) q/2} = \gamma \frac{1 + nq - q\tilde{q}^2 [\beta n + (1 - \beta n)/(2p_0)]}{\tilde{q}^2 - n^2 q^2 (1 - \tilde{q}^2)}. \quad (\text{A11})$$

In the case the frequency variation of the lateral contraction can be neglected, i.e. $\nu(\omega) = \nu = \nu_0$, $m = n = \beta = \gamma = 1$ (and $p_0 = p$) and (A10) reads

$$\frac{M_{zz}(\mathbf{q}, \omega)}{E_r(\omega) q/2} = \frac{(4p_0 - 1) \tilde{q} - \bar{q} (1 - \tilde{q}^2)}{4p_0^2 - \tilde{q}^2 (1 - 2p_0)^2 + \bar{q}^2 (1 - \tilde{q}^2)}, \quad (\text{A12})$$

whereas for the case of free-standing slab (A11) simplifies to

$$\frac{M_{zz}(\mathbf{q}, \omega)}{E_r(\omega) q/2} = \frac{1 + q - q\tilde{q}^2}{\tilde{q}^2 - q^2 (1 - \tilde{q}^2)}, \quad (\text{A13})$$

corresponding to the classical results[4].

2. Limiting cases for the coated half-space

For the case of a bulk $[E_b(\omega), \nu_b(\omega), E_{rb}(\omega) = E_b/(1 - \nu_b^2)]$ coated with a layer $[E(\omega), \nu(\omega), E_r(\omega) = E/(1 - \nu^2)]$ of thickness d , by using (11) we have (after some manipulation)

$$\frac{M_{zz}(\mathbf{q}, \omega)}{E_r(\omega) q/2} = n \frac{c_1 n_0 + c_2 (n_1 M p^2 + 2n_2 m p + n_3 \epsilon_p m^2)}{c_1 d_0 + c_2 (d_1 M p^2 + 2d_2 m p + d_3 \epsilon_p m^2)}, \quad (\text{A14})$$

where

$$\begin{aligned} c_1 &= 4Mp^2 \epsilon_e (\tilde{q} (1 + mq) + nq) + (2Mp - m\epsilon_p) (2p + mq\tilde{q}) \\ &\quad + \epsilon_e \epsilon_p [(m\epsilon_p - 4Mp) mq\tilde{q} - (2mp) (nq + \tilde{q})] \\ c_2 &= -4Mp^2 \epsilon_e (\tilde{q} (1 + mq) + nq) + (2Mp - m\epsilon_p) \gamma (2p\tilde{q} - m(q + \tilde{q})) \\ &\quad + \epsilon_e \epsilon_p [(2Mp) mq\tilde{q} + ((2Mp - m\epsilon_p) + 2mp) (nq + \tilde{q})], \end{aligned}$$

whereas

$$\begin{aligned} n_0 &= -2Mp\epsilon_e (2p - \epsilon_p) [mq\tilde{q} - 2p] \\ &\quad + (2Mp - m\epsilon_p) n \left[\epsilon_e (2p - \epsilon_p) \gamma ((2p - m) \beta \tilde{q} - mq) + \frac{n}{m} 4p\tilde{q} - n(q + \tilde{q}) \right] \\ n_1 &= 8\epsilon_e (\beta \gamma p \tilde{q} + p - \epsilon_p) \\ n_2 &= -2Mp\epsilon_e [q(\gamma + \tilde{q}) + \beta \gamma \tilde{q}] + \epsilon_e \epsilon_p [1 - 2\tilde{q}\epsilon_e \epsilon_p (\beta \gamma p - Mq)] - Mq\tilde{q} \\ n_3 &= 2\gamma p \epsilon_e (q + \beta \tilde{q}) + q\tilde{q} (1 - \epsilon_e \epsilon_p) \end{aligned}$$

and

$$\begin{aligned} d_0 &= 2Mp\epsilon_e (2p - \epsilon_p) m (nq - \tilde{q}) \\ &\quad + (2Mp - m\epsilon_p) [\epsilon_e (2p - \epsilon_p) nq\tilde{q} + \gamma mq\tilde{q} - 2p] \\ d_1 &= 4n\tilde{q} (\beta + nq\epsilon_e) \\ d_2 &= 2M\epsilon_e (nq - \tilde{q}) (p - \epsilon_p) - n\tilde{q}\epsilon_p (\beta + nq\epsilon_e) + nM (q - \beta \tilde{q}) \\ d_3 &= n (\beta \tilde{q} - q) - \epsilon_e \epsilon_p (\tilde{q} - nq), \end{aligned}$$

with $\varepsilon_e = E(\omega)/E_b(\omega)$, $\varepsilon_p = p(\omega)/p_b(\omega)$, and $M = \varepsilon_p/\varepsilon_{0p} = m/(p_b/p_{0b})$, with m , n , γ and β given by (A5). In the limit where the frequency variation of the lateral contraction can be neglected, i.e. $\nu(\omega) = \nu_0$ and $\nu_b(\omega) = \nu_{0b}$ (resulting into $\beta = \gamma = m = n = M = 1$) we obtain the classical result [4]

$$\frac{M_{zz}(\mathbf{q}, \omega)}{E_r(\omega) q/2} = \frac{n_1 \sinh(2q) + 8\alpha p^2 \cosh(2q) + n_2 q}{8p^2 [\alpha \sinh(2q) + 1] + n_1 [\cosh(2q) - 1] - n_2 q^2}, \quad (\text{A15})$$

where this time

$$\begin{aligned} n_1 &= 8\varepsilon_e p^2 + 4p\varepsilon_{pe}(\varepsilon_e - 1) - \varepsilon_{pe}^2 \\ n_2 &= 2\varepsilon_{pe}(4\varepsilon_e p - \varepsilon_{pe}) \end{aligned}$$

and $\varepsilon_{pe} = \varepsilon_e \varepsilon_p - 1$ [$\varepsilon_e = E(\omega)/E_b(\omega)$, $\varepsilon_p = p/p_b$, as before].

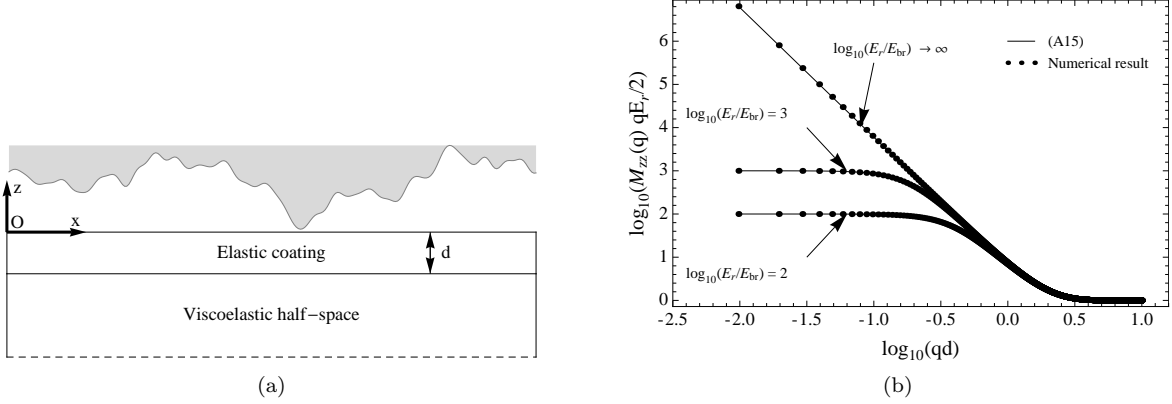


Figure 15. a) Schematic of an elastic coating bonded onto an elastic half space, b) real part of the dimensionless surface response $M_{zz}(q) q E_{r0}/2$ as a function of the wave number qd ($q_0 = 2\pi/L_0$). The bulk is characterized by a reduced elastic modulus E_{rb} , whereas the coating by E_r . For the dimensionless coating thickness $q_0 d = 0.0195$, and for different values of E_r/E_{rb} . The continuous line is from A15, whereas dots are from the application of (11).

In Fig. 15, for (a) an elastic coating bonded onto an elastic half space, we show the (b) real part of the dimensionless surface response $M_{zz}(q) q E_{r0}/2$ as a function of the wave number qd ($q_0 = 2\pi/L_0$). The bulk is characterized by a reduced elastic modulus E_{rb} , whereas the coating by E_r . For the dimensionless coating thickness $q_0 d = 0.0195$, and for different values of E_r/E_{rb} . The continuous line is from (A15), whereas dots are from the application of (11), confirming the validity of the numerical tool.

Finally, in Fig. 16 for an elastic coating bonded onto a viscoelastic half space [see e.g. Fig. 15(a)], we show the real part of the dimensionless surface response $M_{zz}(q, \omega) q E_{r0}/2$ (with $\omega = qv$ and $q_y = 0$) as a function of the wave number q/q_0 ($q_0 = 2\pi/L_0$). The bulk is characterized by a single relaxation time $\tau = L_0/v_1 = 0.01s$ and $E_{\infty}/E_0 = 10$, with $\nu_b(\omega)^{-1} = \nu_{\infty b}^{-1} + \nu_1^{-1}/(1 + \omega^2 \tau^2)$. The coating has an elastic modulus $E = E_{0b}$ and $\nu = 0.49$. The dashed lines [$\nu_b = \nu_b(\omega)$] correspond to (A14), whereas the solid lines [given by considering $\nu_b = \nu_{0b}$] are for (A15). In (a) the dimensionless coating thickness $q_0 d$ is $9.4 \cdot 10^{-3}$, whereas in (b,c) the $q_0 d$ belongs to $[0, 9.4, 63, 190, \infty] \cdot 10^{-3}$. The sliding velocity is set to $v = 0.02v_1$. Moreover, in Fig. 16(a) and 16(b) we have adopted $\nu_{\infty b} < \nu_{0b}$, and inversely for Fig. 16(c).

We note first in Fig. 16(b) that the effective compliance of the composite with $\nu_b = \nu_b(\omega)$ (dashed lines) shows a global maximum, which is close to a range of frequencies where $\nu_b(\omega)$ moves from the rubbery (0.49) to the glassy (0.3) region. Moreover such a maximum, which is even larger than 1, increases for decreasing coating thickness. Far from the previous stationary point (in a log-scale), i.e. at small ($qv \rightarrow 0$) and large ($qv \rightarrow \infty$) roughness frequencies, the compliance converges to the corresponding curve for the frequency-independent ν_b . This can be justified as follows. For $qv \rightarrow 0$ ($qv \rightarrow \infty$), the asperities do only probe the rubbery bulk (coating) of the composite, resulting in $M_{zz} \rightarrow q E_{r0}/2$. For intermediate wavelengths, the bulk undergoes a transition from an incompressible stage in the rubbery ($\nu_b = 0.49$) regime, to a compressible stage in the glassy ($\nu_b = 0.3$) region. Since a continuity of lateral contraction must hold at the coating/bulk interface, the more qv increases the less is the lateral contraction (i.e. ν_b decreases) coped with an increased (viscoelastic) stiffening, resulting that the composite must show a more compliant response in order to match such an interface lateral contraction. Of course the opposite holds for Fig. 16(c). Thus, we observe that neglecting the frequency-dependence of the Poisson's ratio can qualitatively and quantitatively affect the contact mechanics predictions, given the large differences in the effective surface response in a range of frequencies

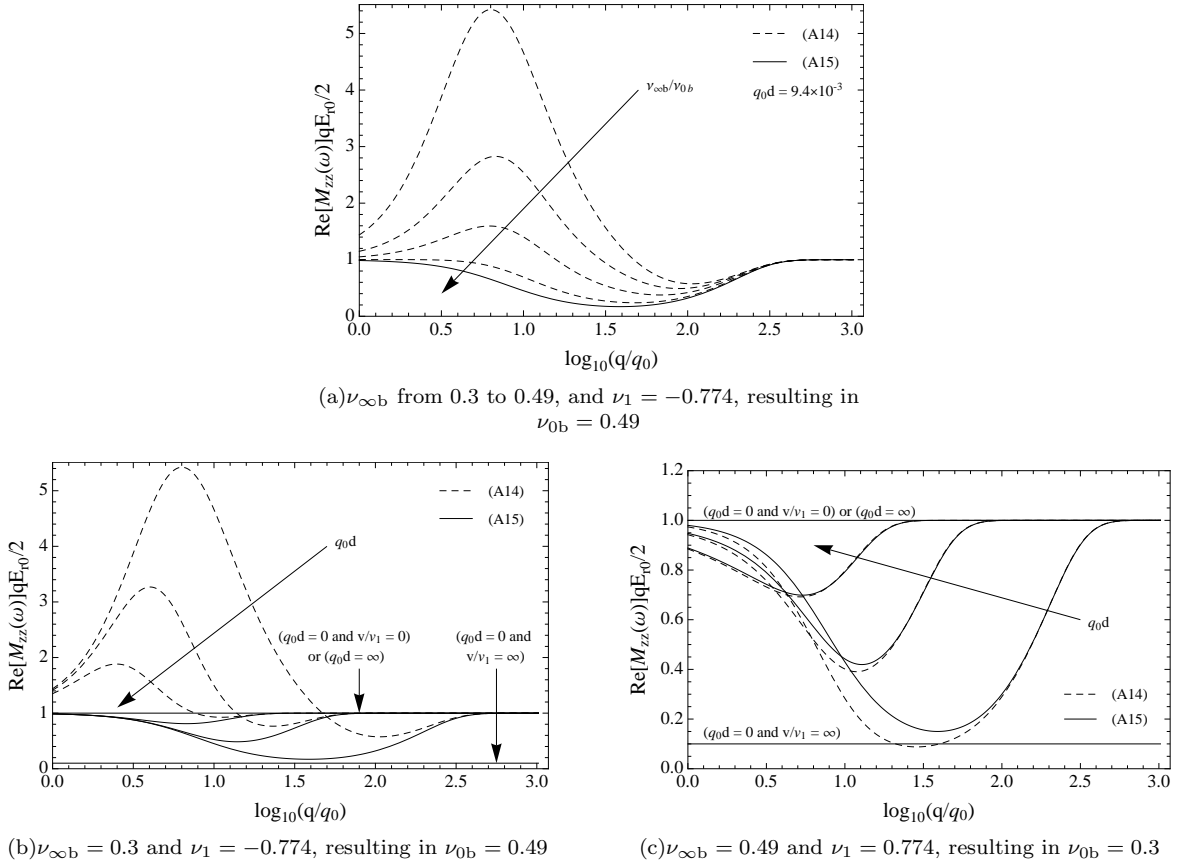


Figure 16. Real part of the dimensionless surface response $M_{zz}(q, \omega) q E_{r0}/2$ (with $\omega = qv$ and $q_y = 0$) as a function of the wave number q/q_0 ($q_0 = 2\pi/L_0$), for an elastic coating bonded onto a viscoelastic half space [see e.g. Fig. 15(a)]. The bulk is characterized by a single relaxation time $\tau = L_0/v_1 = 0.01s$ and $E_\infty/E_0 = 10$, with $\nu_b(\omega)^{-1} = \nu_{\infty b}^{-1} + \nu_1^{-1}/(1 + \omega^2\tau^2)$. The coating has an elastic modulus $E = E_{0b}$ and $\nu = 0.49$. The dashed lines $[\nu_b = \nu_b(\omega)]$ correspond to (A14), whereas the solid lines $[\nu_b = \nu_{0b}]$ are for (A15). In (a) the dimensionless coating thickness q_0d is $9.4 \cdot 10^{-3}$, whereas in (b,c) the q_0d belongs to $[0, 9.4, 63, 190, \infty] \cdot 10^{-3}$.

which cannot be established a priori, even for a simple composite arrangement such as the one previously adopted.

Appendix B: General theory for the finite thickness slab with continuously graded rheological properties

In this section we derive the \mathbf{M} matrix for a composite characterized by continuously-graded rheological properties. In particular, by differentiating (10) we obtain

$$[\mathbf{M} + d\mathbf{M}] [\mathbf{M}_1 + (1 + \alpha dz) \mathbf{M}_2 \mathbf{M}] = [\mathbf{M}_3 + (1 + \alpha dz) \mathbf{M}_4 \mathbf{M}], \quad (\text{B1})$$

with $\alpha(\omega, z) = E_r(\omega, z)^{-1} \partial E_r(\omega, z) / \partial z$. This results in the following set of non-linear differential equations

$$\begin{aligned} \frac{pq}{m^2 p_0^2} &= M'_{11} - \alpha M_{11} + M_{11}^2 \delta_x + M_{11} (M_{12} + M_{21}) \delta_{xy} + M_{12} M_{21} \delta_y \\ &\quad + i q_x \delta_1 (M_{31} - M_{13}) + q \delta_2 M_{13} M_{31} \\ 0 &= M'_{12} - \alpha M_{12} + M_{11} M_{12} \delta_x + (M_{11} M_{22} + M_{12}^2) \delta_{xy} + M_{12} M_{22} \delta_y \\ &\quad + i q_x \delta_1 (M_{32} - M_{13}) + q \delta_2 M_{13} M_{32} \\ 0 &= M'_{13} - \alpha M_{13} + M_{11} M_{13} \delta_x + (M_{11} M_{23} + M_{12} M_{13}) \delta_{xy} + M_{12} M_{23} \delta_y \\ &\quad + i q_x \delta_1 M_{33} + q \delta_2 M_{13} M_{33} + \delta_3 (M_{11} q_x + M_{12} q_y) \end{aligned}$$

$$\begin{aligned}
0 &= M'_{21} - \alpha M_{21} + M_{11}M_{21}\delta_x + (M_{11}M_{22} + M_{21}^2)\delta_{xy} + M_{21}M_{22}\delta_y \\
&\quad + i\delta_1(M_{31}q_y - M_{23}q_x) + q\delta_2M_{23}M_{31} \\
\frac{pq}{m^2p_0^2} &= M'_{22} - \alpha M_{22} + M_{12}M_{21}\delta_x + M_{22}(M_{12} + M_{21})\delta_{xy} + M_{22}^2\delta_y \\
&\quad + iq_y\delta_1(M_{32} - M_{23}) + q\delta_2M_{23}M_{32} \\
0 &= M'_{23} - \alpha M_{23} + M_{13}M_{21}\delta_x + (M_{13}M_{22} + M_{21}M_{23})\delta_{xy} + M_{22}M_{23}\delta_y \\
&\quad + iq_y\delta_1M_{33} + q\delta_2M_{23}M_{33} + (M_{21}q_x + M_{22}q_y)\delta_3 \\
0 &= M'_{31} - \alpha M_{31} + M_{11}M_{31}\delta_x + (M_{11}M_{32} + M_{21}M_{31})\delta_{xy} + M_{21}M_{32}\delta_y \\
&\quad - iq_x\delta_1M_{33} + q\delta_2M_{31}M_{33} - (M_{11}q_x + M_{21}q_y)\delta_3 \\
0 &= M'_{32} - \alpha M_{32} + M_{12}M_{31}\delta_x + (M_{12}M_{32} + M_{22}M_{31})\delta_{xy} + M_{22}M_{32}\delta_y \\
&\quad - iq_y\delta_1M_{33} + q\delta_2M_{32}M_{33} - (M_{12}q_x + M_{22}q_y)\delta_3 \\
-\frac{nq(m-2p)}{2m^3p_0^2} &= M'_{33} - \alpha M_{33} + M_{13}M_{31}\delta_x + (M_{13}M_{32} + M_{23}M_{31})\delta_{xy} + M_{23}M_{32}\delta_y \\
&\quad + q\delta_2M_{33}^2 + [(M_{31} - M_{13})q_x + (M_{32} - M_{23})q_y]\delta_3
\end{aligned}$$

where we have defined

$$\begin{aligned}
\delta_1 &= \frac{\beta(2p_0 - 1) + 1}{2p_0}, \quad \delta_2 = m \left(\frac{1}{n} - 1 \right), \quad \delta_3 = \frac{i\gamma(p_0 - 1)}{p_0} \\
\delta_x &= \frac{(n+1)q_x^2 + pq_y^2}{q}, \quad \delta_y = \frac{(n+1)q_y^2 + pq_x^2}{q}, \quad \delta_{xy} = \frac{(n+1-p)q_xq_y}{q},
\end{aligned}$$

and with BCs $M(\omega, z=0)$ given by (12).

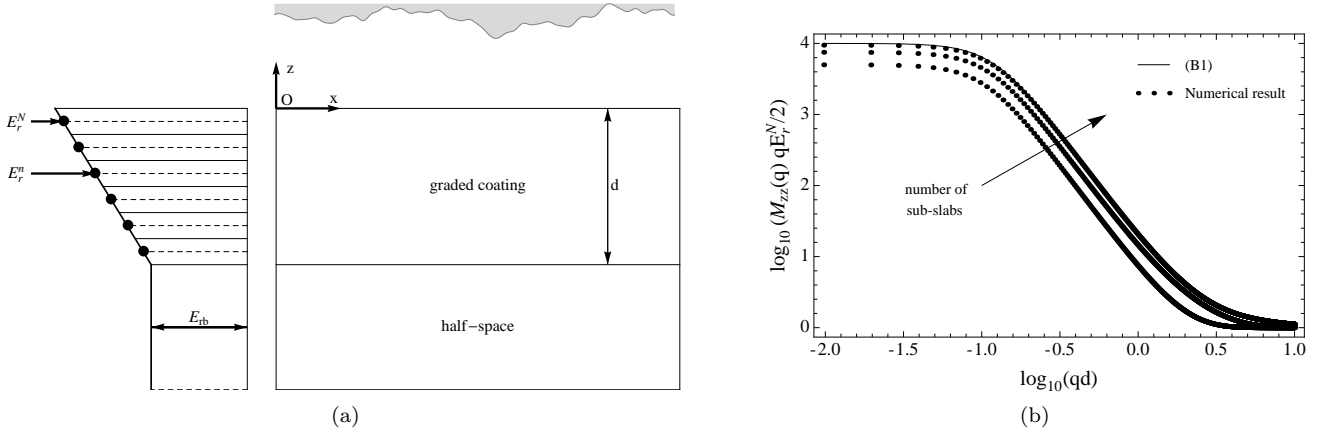


Figure 17. a) Schematic of a graded elastic coating bonded onto an elastic half space, b) dimensionless surface response $M_{zz}(q) q E_{r0}/2$ (with $q_y = 0$) as a function of the wave number dq ($q_0 = 2\pi/L_0$). The coating is characterized by a reduced elastic modulus $E_r(z) = E_{rb} [(1 + 10^4(z+d)/d)]$, whereas the bulk by E_{rb} . For a dimensionless coating thickness $q_0d = 0.0195$ and $L_0 = 0.1$. In (b) the continuous line is from B1, whereas the dots are obtained from the application of (11) to a linear discretization of the coating $E_r(z)$ into a stepwise composite of 1, 2 and 9 layer sub-divisions.

In Fig. 17 and 18 we show b) the dimensionless surface response $M_{zz}(q) q E_{r0}/2$ (with $q_y = 0$) as a function of the wave number dq ($q_0 = 2\pi/L_0$) for a graded elastic coating bonded onto an elastic half space with reduced elastic modulus $E_r(z) = E_{rb} [(1 + 10^4(z+d)/d)]$ and $E_r(z) = E_{rb} [1 + (z+d)/d + \sin(2\pi(z+d)/d)]$, respectively. In the figures the continuous line is from B1, whereas the dots are obtained from the application of (11) to a linear discretization of the coating $E_r(z)$ into a stepwise composite with different layer sub-divisions (see figures caption). We observe that, whilst for the linear graded rheology the surface response of the stepwise composite converges to the continuously-graded in relatively few sub-layer divisions, for the case of Fig. 18 about a two orders of magnitude refined discretization is required to reach convergence, i.e. for complex rheological laws (as e.g. for biological applications)

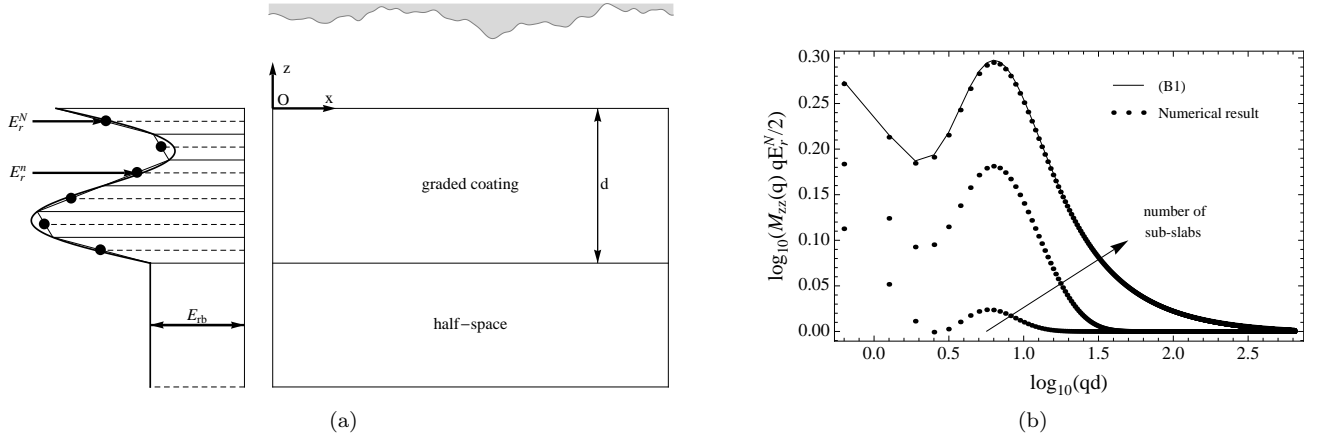


Figure 18. a) Schematic of a graded elastic coating bonded onto an elastic half space, b) dimensionless surface response $M_{zz}(q) q E_{r0}/2$ (with $q_y = 0$) as a function of the wave number dq ($q_0 = 2\pi/L_0$). The coating is characterized by a reduced elastic modulus $E_r(z) = E_{rb} [1 + (z + d)/d + \sin(2\pi(z + d)/d)]$, whereas the bulk by E_{rb} . For a dimensionless coating thickness $q_0 d = 0.0195$ and $L_0 = 0.1$. In (b) the continuous line is from (B1), whereas the dots are obtained from the application of (11) to a linear discretization of the coating $E_r(z)$ into a stepwise composite of 4, 9 and 299 layer sub-divisions.

the adoption of (B1) should be computationally preferred to (11).

-
- [1] Lorenz B., Pyckhout-Hintzen W., and Persson B.N.J. Master curve of viscoelastic solid: Using causality to determine the optimal shifting procedure, and to test the accuracy of measured data. *Polymer (United Kingdom)*, 55(2):565–571, 2014.
 - [2] Persson B.N.J. Theory of rubber friction and contact mechanics. *Journal of Chemical Physics*, 115(8):3840–3861, 2001.
 - [3] Persson B.N.J. Contact mechanics for randomly rough surfaces. *Surface Science Reports*, 61(4):201–227, 2006.
 - [4] Persson B.N.J. Contact mechanics for layered materials with randomly rough surfaces. *Journal of Physics: Condensed Matter*, 24(9):095008, 2012.
 - [5] B. Derjaguin. Untersuchungen ber die reibung und adhsion, iv. *Kolloid-Zeitschrift*, 69(2):155–164, 1934.
 - [6] J. A. Greenwood and J. B. P. Williamson. Contact of nominally flat surfaces. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 295(1442):300–319, 1966.
 - [7] Scaraggi M. and Persson B.N.J. Rolling friction: Comparison of analytical theory with exact numerical results. *Tribology Letters*, 55(1):15–21, 2014.
 - [8] Scaraggi M. and Persson B.N.J. Theory of viscoelastic lubrication. *Tribology International*, 72:118–130, 2014.
 - [9] M. Mokhtari and D.J. Schipper. Existence of a tribo-modified surface layer of br/s-sbr elastomers reinforced with silica or carbon black. *Tribology International*, 2014. Article in Press.
 - [10] M. Paggi and G. Zavarise. Contact mechanics of microscopically rough surfaces with graded elasticity. *European Journal of Mechanics, A/Solids*, 30(5):696–704, 2011.
 - [11] B.N.J. Persson and M. Scaraggi. Theory of adhesion: Role of surface roughness. *Journal of Chemical Physics*, 141(12), 2014.
 - [12] M. Scaraggi. 2015.
 - [13] M. Scaraggi and B.N.J. Persson. Friction and universal contact area law for randomly rough viscoelastic contacts. *Journal of Physics Condensed Matter*, 27(10), 2015.
 - [14] JAN-KE SCHWEITZ and LEIF HMAN. Chapter 9 - mild wear of rubber-based compounds. In Klaus Friedrich, editor, *Friction and Wear of Polymer Composites*, volume 1 of *Composite Materials Series*, pages 289 – 327. Elsevier, 1986.

